Asset Pricing in a Lucas Economy with Recursive Utility Heterogeneous Agents

Hervé Roche*
Departmento de Administración
Instituto Tecnológico Autónomo de México
Av. Camino a Santa Teresa No 930
Col. Héroes de Padierna
10700 México, D.F.
E-mail: hroche@ciep.itam.mx

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Abstract

We extend the Lucas economy (1978) to the case of preferences a la Kreps-Porteus (1978) when the dividend (fruit) process follows a geometric Brownian Motion. In the representative agent case, the equilibrium price reveals a two-stage mechanism. First, the risk averse agent adjusts downward the average growth rate of dividends to incorporate uncertainty. Then, the effect of the intertemporal elasticity of substitution (I.E.S.) depends on the sign of this adjusted growth rate of dividends. In agreement with Hall (1988), the paper illustrates the key role played by I.E.S. on the equilibrium price. The extension to the heterogeneous agents case allows us to analyze the wealth distribution and its effects on the equilibrium price. Individuals who either highly value the future or who are both willing to substitute consumption over time and display low risk aversion will asymptotically accumulate all the wealth in the economy. Wealth concentration leads to an increase in the equilibrium price.

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1. INTRODUCTION

A huge literature in financial economics and macroeconomics has clearly pointed out the limitations of the Expected Utility Theory in its inability to disentangle individuals' willingness to alter their consumption levels across time and individuals' attitudes towards risk. At an empirical level, when using isoelastic utility functions, estimations yield an unreasonably high value for the coefficient of relative risk aversion.

The objective of this paper is to twofold. First, its sheds some additional light on the effects of risk aversion and intertemporal elasticity of substitution (I.E.S.) on asset prices in a general equilibrium setting. Second, the heterogeneous agents framework highlights the role played by the discount rate, the I.E.S. and risk aversion on the wealth sharing rules and the asset equilibrium price.

The idea of introducing preferences allowing distinction between I.E.S. and risk aversion is not new in macroeconomics and finance. We now review the existing literature on the topic and present the main findings of the paper.

1.1. Related Literature

Epstein and Zin (1989) building on the work of Kreps and Porteus (1978) construct a discrete time framework and point out the advantages of using recursive utility. First of all, an important property of the recursive preferences is that they exhibit intertemporal consistency. In addition, such preferences take into account the agents attitude with respect to the temporal resolution of uncertainty. Finally, recursive preferences disentangle the effects of I.E.S. and risk aversion. Agents' preferences are defined in a recursive way as follows. At period t, the agent determines the certainty equivalent $m(\tilde{V}_{t+1} \mid \mathcal{F}_t)$ of the next period lifetime utility V_{t+1} given the information structure $\mathcal{F}_t$. In a second stage, she combines the certainty equivalent m with the current period level of consumption c_t via an aggregator W such that $V_t = W\left(c_t, m(\tilde{V}_{t+1} \mid \mathcal{F}_t)\right)$. The certainty equivalent m encapsulates risk aversion whereas the aggregator W embodies the intertemporal substitution of consumption. These preferences have been used for instance to try to solve the equity premium puzzle (see Weil 1989 and 1990). In two papers, Duffie and Epstein (1992a) and (1992b) present stochastic differential utility which adapts the concept of recursive utility to a continuous time setting and they explore its implications for asset prices. Shroder and Costias (1999) develop a gradient approach to characterize optimal portfolio allocations and consumption plans that maximize stochastic differential utility. For complete markets, Dumas, Uppal and Wang (2000), using the concept of variational utility[1], construct a social planner value function when agents have recursive utility functions. They show that the efficient allocation is also the solution of the decentralized problem. El Karoui, Peng and Quenez (1997) provide a theoretical treatment of backward stochastic differential equations (BSDE) for a finite horizon with some applications to financial economics.

Some of the central issues of the first part of this paper are related to the work by Epstein (1988) and Naik (1994). Epstein investigates the impact of preferences on equilibrium asset prices in a modified Lucas (1978) economy where a unique good can be produced by several different types of trees with i.i.d. output. Naik examines the impacts of adjustment costs and changes in risks on aggregate stock prices. Both authors identify the I.E.S. as determining the direction of the effect of changes in risks of the equilibrium asset price.

The second part of the paper complements the work by Dumas (1989) and Wang (1996). Dumas considers the case of a dynamically complete economy where two heterogenous investors trade a riskless bond and a risky security. Agents differ by their CRRA. The author analyses the optimal portfolio and consumption policies, the equilibrium interest rate and the dynamics of the equilibrium wealth sharing rules. Wang derives a closed-form solution when one investor has a logarithmic utility function and the second investor has a square-root utility function. Another related work on heterogenous agents and asset pricing is the paper by Constantinides and Duffie (1996). They construct a model with heterogenous consumers having common separable preferences but experiencing non-insurable idiosyncratic shocks. They show the existence of an equilibrium with no trade. They show how the Euler Equation must be modified to incorporate the presence of idiosyncratic shocks. In particular, a term, representing the cross-sectional variance of the distribution of the individual consumer's consumption growth, needs to be added. The main result is that ignoring idiosyncratic risks leads to an underestimation or overestimation of a security excess return. In this paper, only a risky asset can be traded by N heterogenous investors and we investigate how differences in preference characteristics affect the asset equilibrium price and ownership distribution across time.

At an empirical level, allowing distinction between I.E.S. and risk aversion in agents' preferences can be useful to understand the limits implied by Expected Utility Theory in fitting time-series behavior of consumption and asset returns. Epstein and Zin (1991) using monthly U.S. data on consumption and returns from 1959-1986 obtain a value of I.E.S. less than unity (corroborating Hall's estimate, 1988). Nevertheless, they stipulate that the results are sensitive to the choice

[1] This concept was introduced by Geoffard (1996) in a deterministic framework.

of consumption measure. Kandel and Stambaugh (1991) show that risk aversion plays an important role in accessing the means of both equity returns and interest rate. On the other hand, the volatility of equity returns is determined primarily by the I.E.S., a low I.E.S. implying a high volatility. Kocherlakota (1990) argues that the real problem for an econometrician lies in disentangling I.E.S. and the discount factor. He concludes that "the link between the CRRA and the I.E.S. in the standard preferences is not the cause of their empirical failure". A recent paper by Schwartz and Torous (2000) argues that data used in some previous studies that try to disentangle I.E.S. and risk aversion were inappropriate. Using U.S. term structure data over the period 1964-1997, they find the I.E.S. to be equal to 0.226 over 1964-1997 (0.11 over 1979-1997), the CRRA coefficient to be equal to 5.65 over 1964-1997 (8.83 over 1979-1997) and the discount factor to be around 0.989 for both time periods.

1.2. Results

The main contribution of the paper is to clarify the effects of risk aversion and I.E.S. on equilibrium price. We consider a Lucas (1978) tree economy with identical trees (of measure one) whose output is correlated over time and assume a constant average growth rate of dividends. In the representative agent case, we are able to get a closed form solution which provides some new insights on the role played by risk aversion and I.E.S. in the formation of the asset equilibrium price. The equilibrium price reveals a two stage mechanism concerning the effects of risk aversion and willingness to substitute consumption over time. First, the agent adjusts downward the average growth rate of dividends with a (negative) premium proportional to her degree of risk aversion and the magnitude of risk. Then, the effect of this effective growth rate of dividends depends on the I.E.S. Since there is no storage, if the agent's willingness to substitute present consumption with future consumption is high (low) enough, namely I.E.S. greater (less) than *unity*, an increase in the effective growth rate drives up (down) the price of the tree. In the second stage, the agent looks at the sign of the effective growth rate of dividends. If the latter is positive (negative), the higher I.E.S., the greater (lower) is the equilibrium price.

The extension to the heterogeneous agents case allows us to trace out the evolution of the ownership distribution across time and its effects on the equilibrium price. We find that agents can be ranked using a one-dimensional parameter θ, depending on individuals' discount rate for the future, I.E.S. and risk aversion. The agents, whose value of θ is small, are the agents who either are very patient (low discount rate for future) or combine a low risk aversion with a high willingness to substitute consumption across time. The agent displaying the lowest value for θ sees her consumption and ownership rising with time and will ultimately become the only capitalist in the economy. Wealth concentration leads to an increase in the equilibrium price of the risky asset.

The paper is organized as follows. Section 2 describes the economic setting and provides some insights on the role of the I.E.S. and risk aversion in the determination of the equilibrium price of an asset. Section 3 extends the analysis to the heterogeneous agents economy and focuses on the wealth (ownership) distribution among agents and its effects on the equilibrium price. Section 4 concludes. Proofs of all results are collected in the appendix.

2. THE ECONOMIC SETTING

We consider an economy *à la* Lucas (1978). Time is continuous and the economy is populated with a continuum of measure 1 of identical agents who live forever. The main innovation of the paper lies in the introduction of a class of homothetic recursive utility functions a la Kreps-Porteus (1978).

2.1. Information Structure

Uncertainty arises from the productive sector.

It is modeled by a probability space $(\Omega, \mathcal{F}, P)$ on which is defined a *one* dimensional Brownian Motion w. A state of nature ω is an element of Ω. $\mathcal{F}$ denotes the tribe of subsets of Ω that are events over which the probability measure P is assigned. The information structure is given by a standard filtration $\mathbb{F} = \{\mathcal{F}_t,\ t \in \mathbb{R}_+\}$ satisfying the usual conditions (increasing, right-continuous, augmented and with $\mathcal{F}_0$ being trivial. That is $\mathcal{F}_t$ is the σ-algebra generated by $\{w(s); 0 \leqslant s \leqslant t)\}$ and augmented. At time t, the information set is $\mathcal{F}_t$: individuals can learn the true state of nature by observing the sample path (realization) of w over time. The filtration $\mathbb{F}$ represents how information is revealed over time. All the processes considered in the paper are progressively measurable with respect to $\mathbb{F}$ and all identities involving random variables (stochastic processes) should be understood to hold $P-a.s.$ $((\lambda \times P) - a.e.$, where λ denotes the Lebesgue measure on $\mathbb{R}_+$).

Note that in this economy markets are not (dynamically) complete since there is *one* source of uncertainty and only *one* stock can be traded.

2.2. Preferences

Preferences are represented by a stochastic differential utility characterized by a pair of primitives (f, A) called the (un)normalized aggregator. The representative agent maximizes her lifetime utility

$$V(t) = E_t\left(\int_t^{\infty}\left(f(c(s), V(s)) + \frac{1}{2}A(V(s))\left\|\sigma_V(s)\right\|^2\right)ds\right) \tag{2.1}$$

where c is a consumption process, E_t denotes the conditional expectation given $\mathcal{F}_t$ and σ_V is a progressively measurable square integrable process. Equivalent to (2.1), given a consumption process c, the utility process V satisfies the stochastic differential equation

$$dV(t) = -\left(f(c(t), V(t)) + \frac{1}{2}A(V(t))\left\|\sigma_V(t)\right\|^2\right)dt + \sigma_V(t)dw(t). \tag{2.2}$$

The attitude to risk is encapsulated in the function A and broadly speaking, "the more negative is A, the more risk averse is the agent." We use stochastic differential utility a la Kreps-Porteus (1978). The aggregator is given by

$$f(c, v) = \frac{\beta}{\rho}\frac{c^{\rho} - v^{\rho}}{v^{\rho-1}},\; A(v) = \frac{\alpha - 1}{v}$$

with $\beta > 0$, $0 \neq \rho \leqslant 1$ and $0 \neq \alpha \leqslant 1$. The discount rate of future is β, the intertemporal elasticity of substitution is $s = \frac{1}{1-\rho}$, and risk aversion is captured by α. For instance, when $\alpha = 1$, the agent is risk neutral. When $\alpha = \rho$, we are in the standard framework of Expected Utility and the coefficient of relative risk aversion is $1 - \alpha$. These preferences are time consistent and the individual is not indifferent to the timing of resolution of uncertainty. In particular, if $s > \frac{1}{1-\alpha}$, the agent prefers early resolution and if $s < \frac{1}{1-\alpha}$, late resolution.

As shown in Duffie and Epstein (1992b), this program is equivalent to maximizing a program using a normalized aggregator $(\overline{f}, \overline{A})$ such that $\overline{A} \equiv 0$. The corresponding normalized aggregator is:

$$\overline{f}(c, v) = \frac{\beta}{\rho}\frac{c^{\rho} - (\alpha v)^{\frac{\rho}{\alpha}}}{(\alpha v)^{\frac{\rho}{\alpha}-1}}\;.$$

The existence and uniqueness of the objective function V for a given (well behaved) consumption process is shown in Duffie and Lions (1992) using PDE techniques. They prove that under exponential growth conditions for the consumption process, the lifetime utility process V defined by equation (2.2) satisfies a PDE which has a unique solution.

We slightly modify the Duffie-Epstein (1992) formulation of SDU in order to get existence of the consumer maximization problem in an infinite horizon with growing consumption plans.

Define a modified utility function W by

$$W(t) \equiv V(t)e^{-\gamma t}$$

where γ is an arbitrarily large positive number. Notice that maximizing V is equivalent to maximizing W. Given equation (2.2), the function W satisfies

$$dW(t) = -\left(e^{-\gamma t}\overline{f}(c(t), e^{\gamma t}W(t))dt + \gamma W(t)\right) dt + \sigma_V(t)e^{-\gamma t}dw(t).$$

If $\overline{f}(c,v) = \frac{\beta}{\rho}\frac{c^\rho - (\alpha v)^{\frac{\rho}{\alpha}}}{(\alpha v)^{\frac{\rho}{\alpha}-1}}$, then we have the representation

$$W(t) = E_t\left[\int_t^\infty \left(\frac{\beta}{\rho}\frac{c(s)^\rho}{(\alpha W(s))^{\frac{\rho}{\alpha}-1}}e^{-\frac{\gamma\rho}{\alpha}s} - (\alpha\frac{\beta}{\rho} - \gamma)W(s)\right) ds\right].$$

The terms $e^{-\gamma t}$ plays the role of a discount factor and will assure convergence of the integral even when the consumption process grows on average at a positive rate. One can also notice that this formulation is equivalent to redefining the aggregator using the function by $\widetilde{f}(c,v,t) = \frac{\beta}{\rho}\frac{e^{-\frac{\gamma\rho}{\alpha}t}c^\rho}{(\alpha v)^{\frac{\rho}{\alpha}-1}} - (\alpha\frac{\beta}{\rho} - \gamma)v$.

2.3. Technology

There is a continuum of measure 1 of identical trees. Output of each tree y is perishable and follows a geometric Brownian Motion

$$dy(t) = y(t)\left(\mu dt + \sigma dw(t)\right)$$

where $dw(t)$ is the increment of a standard Wierner process, μ represents the average growth rate of output and σ captures the magnitude of the uncertainty.[2]

Assumption

In order to have existence of the SDU, we need to assume some growth restrictions on the output process. In particular we impose the condition $\beta + \rho\left(-\mu + (1-\alpha)\frac{\sigma^2}{2}\right) > 0$.

[2]In a discrete time framework, an equivalent formulation is $y_{t+1} = (1+\mu)y_t + y_t\sigma\varepsilon_{t+1}$, where ε_{t+1} is i.i.d. and normally distributed.

2.4. Representative Agent Problem

Taking the price of a tree p as given, the agent decides how many units of stock z to hold and how much to consume c in order to maximize her lifetime utility. Since there is only *one* asset in the economy, the agent's program can be written

$$\max_c W_t$$

$$\begin{aligned} \text{s.t. } dz(s) &= \frac{1}{p(s)}\left[z(s)y(s)ds - c(s)ds\right], \ z(t) > 0 \text{ given} \\ dy(s) &= y(s)\left(\mu ds + \sigma dw(s)\right) \end{aligned}$$

Transversality Condition: Following Duffie, Epstein and Skiadas (1992 Appendix C), the tranversality condition for this problem can be written

$$\lim_{t\to\infty} e^{-\upsilon t} E\left[W_t\right] = 0 \text{ for some suitable } \upsilon > 0.$$

We now move to the equilibrium properties.

2.5. Equilibrium Analysis

We already mentioned existence and uniqueness of the equilibrium, characterized by the condition $c = y$ or equivalently, $z = 1$. Moreover, we have the following proposition.

Proposition 1. *The equilibrium value function J is given by $J(y,t) = Ke^{-\gamma t}y^{\alpha}$ where K is a positive constant independent of γ but that depends on all the other parameters of the model.*

Proof. See appendix 1. ■

It is easy to check that the transversality condition is met for some $\upsilon > 0$ large enough.

Proposition 2. *The equilibrium price of tree p is a linear function of the current dividend y and is given by*

$$p(y) = \frac{y}{\beta - \frac{s-1}{s}\left(\mu - \frac{(1-\alpha)}{2}\sigma^2\right)}. \tag{2.3}$$

Proof. See appendix 2. ■

We use a utility gradient approach to compute the equilibrium price[3].

[3] For more details on this approach, see Duffie and Skiadas (1994) and Shroder and Skiadas (1999).

Remark 1. *For the standard Expected Utility framework, the above result specializes to*

$$p(y) = \frac{y}{\beta - \alpha\mu + \frac{\alpha(1-\alpha)}{2}\sigma^2}$$

Recall that in this case the coefficient of risk aversion is $1-\alpha$ *and the elasticity of substitution is* $\frac{1}{1-\alpha}$.

The above relationship provides several insights about the dependence of the equilibrium price on the fundamentals of the economy.

When the agent elasticity of substitution is equal to 1, which corresponds to the case of logarithmic felicity function, the agent is myopic and does not care about the future. Thus the equilibrium price is independent of the average growth rate of dividends μ, the relative risk aversion measure α and magnitude of uncertainty σ.

Disentangling the effects of risk aversion and I.E.S., relationship (2.3) shows the following. The agent's attitude with respect to uncertainty is to adjust downward the average growth rate of dividends μ by a factor $\frac{(1-\alpha)}{2}\sigma^2$ increasing with the degree of risk $(1-\alpha)$ aversion and the magnitude of risk σ, so that the effective growth rate is $g = \mu - \frac{(1-\alpha)}{2}\sigma^2$. Now, recall that there is no storage in this economy. When the I.E.S. s is greater than 1, the agent is willing to substitute present consumption for future consumption. To do so, she wants to increase her purchase of securities driving prices up as g goes up. On the contrary, when $s < 1$, she prefers smooth consumption plans. The equilibrium price decreases as g rises as she sells the asset in order to increase current consumption. The effects of the magnitude of the I.E.S. depend on the sign of the effective growth rate of dividends. If g is positive, increasing s drives the equilibrium price up: the agent wants to buy the asset which has a positive growth rate of dividends, taking into account risk. This pushes the price up. The opposite applies if g is negative. Campbell and Viciera (1998) present a partial equilibrium model in which an Epstein-Zin recursive utility representative agent trades a riskless asset and a risky asset having time dependent expected return. They use an analytical approximation for solving the Euler equation and calibrate the model to postwar US stock market data. They show that the I.E.S. only indirectly affects portfolio choice through its effects on the average level of consumption relative to wealth. Their calibration results show that this indirect effect is small. They conclude that the main parameter in determining portfolio choice is the coefficient of risk aversion. Regarding empirical estimations of the I.E.S., Hall (1988) concludes from his empirical analysis that the I.E.S. is "likely not to be above 0.2 and may well be zero". Epstein and Zin (1991), Schwartz and Torous (2000) also find a small but positive I.E.S. It could be interesting to have an estimation of the adjusted dividend growth rate or at least its sign in order to identify the effect of the I.E.S. on the asset equilibrium price.

We now introduce heterogeneity in the economy, allowing agents to differ in their willingness to substitute consumption over time, attitude towards risk and degree of impatience.

3. Heterogeneous Agents

In this section, we extend the analysis to the case of an economy populated with N heterogeneous agents. As before, agent i preferences are represented by a stochastic differential utility characterized by a normalized aggregator $(\overline{f}_i, 0)$ with

$$\overline{f}_i(c_i, v_i) = \frac{\beta_i}{\rho_i} \frac{c_i^{\rho_i} - (\alpha v_i)^{\frac{\rho_i}{\alpha_i}}}{(\alpha v_i)^{\frac{\rho_i}{\alpha_i}-1}} .$$

The triplet (β_i, s_i, α_i), describing agent i discount rate for the future β_i, her I.E.S. s_i, and her attitude towards risk α_i, encapsulates the heterogeneity among agents. We define

$$\theta_i = \beta_i - \frac{s_i - 1}{s_i}\left(\mu - \frac{(1-\alpha_i)}{2}\sigma^2\right) \tag{3.1}$$

and we assume that we can rank the agents in such a way that

$$0 < \theta_1 < \theta_2 < ... < \theta_N.$$

As the parameter θ_i turns out to be a sufficient statistics for encapsulating agents heterogeneity, it plays a crucial role for our analysis.

As already pointed out in the previous section, we need to be careful regarding the existence of the value function when dealing with an infinite horizon and growing consumption plans. To be precise, we should modify the individual value function in the same way as before. However, to keep the exposure simple, we will use stochastic differential utilities as presented in Duffie and Epstein (1992). We now set up the program for agent i.

3.1. Individual Program

Aggregate consumption is exogenously given. The state variables for the individual problem are the current level of dividend y and the wealth (ownership) distribution $Z = (z_1, z_2, ..., z_N)$. Agents can trade their shares and can also sell/buy consumption among each other in exchange for the promise of some future consumption.

Notice that we do not have idiosyncratic shocks in our framework. Hence, uncertainty will affect individual's in a uniform fashion but the magnitude of the effect depends on preference characteristics of each agent. The issue of asset pricing in presence of insurable idiosyncratic risks is tackled in Constantinides and Dufffie (1996).

Given a pair $(y, Z) > 0$ at time t, agent i maximizes her lifetime utility

$$\begin{aligned} \max_{c_i} V_i(y, Z) &= E_t \left[\int_t^\infty \frac{\beta_i}{\rho_i} \frac{c_i^{\rho_i}(s) - (\alpha_i V_i(y(s), Z(s)))^{\frac{\rho_i}{\alpha_i}}}{(\alpha_i V_i(y(s), Z(s)))^{\frac{\rho_i}{\alpha_i}-1}} ds \right] \\ \text{s.t. } dz_k(s) &= \frac{1}{p(s)} \left[z_k(s) y(s) - c_k(s) \right] ds, \; k = 1, 2, ..., N \qquad \text{(Pi)} \\ dy(s) &= y(s) \left(\mu ds + \sigma dw(s) \right) \end{aligned}$$

In addition, we required the following Transversality Condition

$$\lim_{t \to \infty} e^{-\upsilon_i t} E\left[V_i(y(t), Z(t)) \right] = 0 \text{ for some suitable } \upsilon_i > 0.$$

We now define an equilibrium for this economy and investigate its properties.

3.2. Equilibrium Analysis

An equilibrium for this economy consists of

1. N couples of functions (c_i, z_i) that are solutions of agent $i's$ problem (Pi)
2. A price function p
3. Market clearing conditions: The allocations $\{(c_i, z_i)\}_{i=1}^N$ must satisfy

$$\begin{aligned} \sum_{i=1}^N c_i &= y \text{ (goods market)} \\ \sum_{i=1}^N z_i &= 1 \text{ (financial market)} \end{aligned}$$

Remark 2. *Since there is only one asset available in the economy, the non-Ponzi game (or Transversality) condition implies that any agent will have a long position in the asset. To see this, assume that at some date* $\tau > 0$,, *agent* i *has* *a short position, $z_i(\tau) < 0$. Because the marginal utility goes to infinity when consumption goes to zero, agent i always chooses to consume a positive amount, $c_i > 0$. If $z_i(\tau) < 0$, then form agent i budget constraint, it is easy to see that $\dot{z}_i(\tau) < 0$. This implies that agent i holdings decrease after τ. This violates the non Ponzi game condition.*[4] *We conclude that $z_i(t) > 0$, for all date t and for $i = 1, ..., N$.*

The following proposition specifies the structure of an equilibrium.

Proposition 3. *There exists an equilibrium where given some initial conditions (y, Z), agent i's equilibrium value function $V_i^*(y, Z) = \Psi_i(Z)\, y^{\alpha_i}$, the equilibrium price $p(y, Z) = P(Z)y$, the optimal consumption $c_i^*(y, Z) = C_i(Z)y$, the optimal portfolio rule z_i is a deterministic function of time and $((\Psi_i, C_i), P)$ are also deterministic functions of time for $i = 1, 2, ..., N$.*

Proof. See appendix 3. ■

Since there are no idiosyncratic shocks, uncertainty affects agents in a similar fashion, i.e., a positive shock is beneficial for every agent. Though, the magnitude of the effect depends on each agent own characteristics. The issue of the effect of idiosyncratic risks affect is addressed by Constantinides and Duffie (1996). In the sequel, we focus on the properties of the equilibrium described in the above proposition. We now characterize the dynamics of the equilibrium.

Proposition 4. *At time t, for $i = 1, 2, ..., N$, writing agent i's equilibrium value function $V_i^*(y(t), Z(t)) = J_i(t)\, y(t)^{\alpha_i}$, the equilibrium allocation $c_i^*(y(t), Z(t)) = h_i(t)y(t)$ and the equilibrium price $p(y(t), Z(t)) = P(t)y(t)$, agent i's program (Pi) collapses to maximizing the usual separable utility*

$$\begin{aligned}\Phi_i(t) &= \int_t^\infty \beta_i(\alpha_i)^{\frac{\rho_i}{\alpha_i}} h_i^{\rho_i}(s) e^{-\theta_i(s-t)} ds \\ s.t.\ dz_i(s) &= \frac{1}{P(s)}\left[z_i(s) - h_i(s)\right] ds\end{aligned}$$

where $\Phi_i \equiv J_i^{\rho_i/\alpha_i}$.

Corollary 1. *Defining the pseudo riskless interest rate r for a unit dividend tree*

$$r = \frac{\dot{P} + 1}{P}$$

[4] As we will see below, $\lim_{t\to\infty} z_i(t) = 0$ for $i = 2, ..., N$ and $\lim_{t\to\infty} h_1(t) = 1$.

the optimal condition for agent i is

$$\frac{\dot{h}_i}{h_i} = \frac{r - \theta_i}{b_i} \tag{3.2}$$

where $b_i = 1 - \rho_i$. Moreover, the transversality condition is

$$\lim_{t \to \infty} z_i(t)\lambda_i(t) = 0$$

where λ_i is the costate variable associated to z_i.

Proof. See appendix 4. ■.

Using the market clearing condition and the optimal conditions for each agent program, the pseudo riskless interest rate r can be expressed as a weighted average (taking consumption levels as weights) of the θ_i, so

$$r = \frac{\sum_{k=1}^{N} \frac{h_k}{b_k}\theta_k}{\sum_{k=1}^{N} \frac{h_k}{b_k}}$$

3.3. Equilibrium Properties

In this paragraph, we conduct our analysis controlling for the level of output y. Broadly speaking, this is equivalent to shutting down the uncertainty and normalizing the output level to 1.

Some interesting properties of the equilibrium depends on the level of consumption h_1 of agent 1. In particular, when h_1 reaches a sufficiently high level, all other agents decrease their level of consumption and asset holding and the equilibrium price increases. The dynamics of the model depend in particular of the initial ownership distribution Z_0. If Z_0 is such that agent 1 owns a very small number of shares and agent N is the main shareholder in the economy, it will take some time for the economy to reach the state in which agent 1 is the only agent accumulating shares and all the other agents are decreasing their holdings.

The dynamics of consumption and ownership are governed by the level of the pseudo risk-free interest rate r. The following proposition characterizes these results.

Proposition 5. *Given any initial condition (y, Z), we have $\theta_1 < r < \theta_N$ and there exists a finite date T such that for all $t > T$, $r(t) < \theta_2$ and for $t \leqslant T$ (unless*

$T = 0$) and $r(t) \geqslant \theta_2$. This date is characterized by a sufficiently high level of consumption h_1 for agent 1, more specifically

$$\sum_{k=3}^{N} \frac{h_k}{b_k}(\theta_k - \theta_2) \leqslant (\theta_2 - \theta_1)\frac{h_1}{b_1} \tag{3.3}$$

From the relationship (3.2), agent i increases consumption as long as r remains greater than θ_i. Moreover after date T, agent 1 remains the only agent who accumulates shares and increases consumption.

Proof. See appendix 5. ■

Remark 3. *Recall that we have $h_1 < z_1$ and $z_N < h_N$. Condition (3.3) cannot be satisfied if agent 1 does not hold enough shares of the trees. Moreover, if the initial condition is such that agent N owns a lot of shares, it is possible to have $r > \theta_{N-1}$.[5] This implies that all individuals but agent N increase their consumption level at least for a while.*

We now present some interesting properties of the equilibrium price, consumption allocations and ownership. All the proofs are provided in appendix 5.

Equilibrium Price

For all $t > 0$, we have

$$\frac{1}{\theta_N} < P(t) < \frac{1}{\theta_1}$$

and for $t > T$,

$$\begin{array}{l} \dot{P}(t) > 0 \\ \frac{1}{\theta_2} < P(t) < \frac{1}{\theta_1} \\ \lim\limits_{t \to \infty} P(t) = \frac{1}{\theta_1} \end{array}$$

The price is always higher than the value which would prevail if agent N were alone in the economy. Wealth concentration by agent 1 leads to a rise in the equilibrium price until the level which would prevail if agent 1 were alone in the economy. This should not come as a surprise. Agents with high θ prefer early consumption. To achieve this, they are willing to sell some of their shares of the trees to agents preferring late consumption. Hence, initially, the asset equilibrium price is "relatively" low. As time passes, since there is a fixed amount of shares,

[5] More specically, if $z_{N0} > \frac{b_N}{\theta_{N.} - \theta_{N-1.}} \sum_{k=1}^{N-2} \frac{\theta_{N-1.} - \theta_{k.}}{b_k}$, then indeed we have $r(0) > \theta_{N-1}$.

agent 1 has accumulated shares and there are fewer shares available on the market. This drives the equilibrium price up.

Consumption
For $t > 0$, we have

$$\begin{aligned} \dot{h}_1(t) &> 0 \text{ and } \lim_{t\to\infty} h_1(t) = 1 \\ \dot{h}_N(t) &< 0 \end{aligned}$$

For $t > T$,

$$\dot{h}_i(t) < 0 \text{ and } \lim_{t\to\infty} h_i(t) = 0 \text{ for } i = 2, ..., N$$

Portfolio Shares
For all $t > 0$, we have

$$\begin{aligned} \dot{z}_1(t) &> 0 \text{ and } \lim_{t\to\infty} z_1(t) = 1 \\ \dot{z}_N(t) &< 0 \end{aligned}$$

and for $t > T$,

$$\dot{z}_i(t) < 0 \text{ and } \lim_{t\to\infty} z_i(t) = 0 \text{ for } i = 2, ..., N.$$

Agent 1 increases her ownership until (asymptotically) she is the only capitalist in the economy.

Borrowers and Lenders
For all $t > 0$, agent 1 is always a lender and agent N is always a borrower,

$$\begin{aligned} z_1(t) &> h_1(t) \\ z_N(t) &< h_N(t) \end{aligned}$$

and for $t > T$, agents $2, ..., N-1$ become borrowers

$$z_i(t) > h_i(t).$$

We now examine the preference characteristics determine which agent ultimately accumulates all the wealth in economy.

3.4. Preference Characteristics and Wealth Accumulation

In this paragraph, we ask the question: Why do some agents accumulate wealth and other do not? This leads us to analyze the dependence of θ_i on β_i, s_i and α_i. Recall that the agent having the lowest value for θ will ultimately own all the shares.

3.4.1. Effect of the Discount Factor

From relationship (3.1), we have $\frac{\partial \theta_i}{\partial \beta_i} = 1$. Agent i can display a low value for θ_i because she values future a lot (high patience or low β_i).

3.4.2. Effect of I.E.S

From relationship (3.1), we have $\frac{\partial \theta_i}{\partial s_i} = -\frac{1}{s_i^2}\left(\mu - \frac{(1-\alpha_i)}{2}\sigma^2\right)$. If agent i's effective growth rate $g_i = \mu - \frac{(1-\alpha_i)}{2}\sigma^2$ is positive, then she can display a low value for θ_i because she is willing to substitute consumption over time. Note that in order to have a positive effective growth rate, the agent must have a sufficiently low aversion towards risk.

3.4.3. Effects of risk aversion

First of all, recall that when α increases, risk aversion decreases. Once again, using relationship (3.1), we have $\frac{\partial \theta_i}{\partial \alpha_i} = -\frac{s_i - 1}{s_i}\frac{\sigma^2}{2}$. If agent i is willing to substitute consumption over time enough ($s_i > 1$), she can display a low value for θ_i if she has a low risk aversion.

As a summary, we can claim that individuals who either highly value the future or who are both willing to substitute consumption over time and display low risk aversion will accumulate wealth.

4. CONCLUSION

An Expected Utility framework is unable to disentangle the effects and therefore the relative importance of risk aversion and intertemporal elasticity of substitution. For an environment *à la* Lucas (1978), i.e., an endowment economy with (in this case) a single perishable good, this paper illustrates the importance of the I.E.S. on the equilibrium price. Risk aversion only plays a role in accessing a corrected average growth rate of dividends that incorporates uncertainty. Introducing heterogeneity among agents highlights the role played by time preference, risk aversion and I.E.S. in wealth accumulation and its effects on the asset equilibrium price. The two-person problem is a special case of our analysis. In the context of three agents or more, as Dumas (1989) points out, the equilibrium behavior can be different from the two-investor situation. In our framework, we find that the asymptotic behavior of the N-person problem coincides with the two-person problem as far as the wealth sharing rules are concerned. The main result is that patience, willingness to substitute consumption over time combined with a low aversion towards risk contribute to wealth accumulation.

In our setting, uncertainty affects agents in a uniform fashion in some sense since we do not allow for idiosyncratic shocks. A natural extension will be to allow labor or income shocks when individual preferences are represented by stochastic differential utility. In addition, because agents cannot store wealth (fruits are perishable) it would be interesting to introduce a riskless asset (storage technology) to improve consumption smoothing and risk hedging. This will allow us to address issues related to the equity premium. This is left for future research.

5. APPENDIX

5.1. APPENDIX 1

Proof of Proposition 1.

Proof. At the equilibrium, $c(t) = y(t)$ and we know that there is a unique function J such that

$$\begin{aligned} J(y(t),t) &= E_t\left(\int_t^{\infty}\left(\tilde{f}(y(s), J\left(y(s),s\right), s\right) ds\right) \\ \text{s.t. } dy(s) &= y(s)\left[\mu ds + \sigma dw(s)\right] \end{aligned}$$

Let us verify that for all t, $J(y(t),t) = Ke^{-\gamma t}y(t)^{\alpha}$ for some constant K to be determined. Plugging back J into the conditional expectation yields

$$Ke^{-\gamma t}y(t)^{\alpha} = E_t\left(\int_t^{\infty}\left(\frac{\beta}{\rho}\frac{e^{-\frac{\gamma\rho}{\alpha}s}y(s)^{\rho}}{\left(\alpha Ke^{-\gamma s}y(s)^{\alpha}\right)^{\frac{\rho}{\alpha}-1}} - (\alpha\frac{\beta}{\rho} - \gamma)Ke^{-\gamma s}y(s)^{\alpha}\right) ds\right)$$

which leads to the condition

$$Ke^{-\gamma t}y(t)^{\alpha} = E_t\left(\int_t^{\infty} y(s)^{\alpha}e^{-\gamma s}\left(\frac{\beta}{\rho}(\alpha K)^{1-\frac{\rho}{\alpha}} - (\alpha\frac{\beta}{\rho} - \gamma)K\right) ds\right).$$

It follows

$$e^{-\gamma t}y(t)^{\alpha} = \left(\frac{\alpha\beta}{\rho}(\alpha K)^{-\frac{\rho}{\alpha}} - (\alpha\frac{\beta}{\rho} - \gamma)\right)\left(\int_t^{\infty} E_t\left(y(s)^{\alpha}e^{-\gamma s}\right) ds\right).$$

Note that for γ large enough, the quantity $\int_t^{\infty} E_t\left(y(s)^{\alpha}e^{-\gamma s}\right)$ is indeed finite and equal to $\frac{e^{-\gamma t}y(t)^{\alpha}}{\gamma - \alpha\mu + \alpha(1-\alpha)\frac{\sigma^2}{2}}$. This leads to the condition

$$\begin{aligned} \frac{\left(\frac{\alpha\beta}{\rho}(\alpha K)^{-\frac{\rho}{\alpha}} - (\alpha\frac{\beta}{\rho} - \gamma)\right)}{\gamma - \alpha\mu + \alpha(1-\alpha)\frac{\sigma^2}{2}} &= 1 \text{ or equivalently} \\ \gamma + \alpha\left(-\mu + (1-\alpha)\frac{\sigma^2}{2}\right) &= \frac{\alpha\beta}{\rho}(\alpha K)^{-\frac{\rho}{\alpha}} - \alpha\frac{\beta}{\rho} + \gamma \end{aligned}$$

The terms in γ of the LHS and RHS cancel out each other and we finally obtain

$$\beta(\alpha K)^{-\frac{\rho}{\alpha}} = \beta + \rho\left(-\mu + (1-\alpha)\frac{\sigma^2}{2}\right)$$

We can readily conclude that there exists a unique constant K independent of γ and positive given the assumption made on the parameters of the model. ■

5.2. APPENDIX 2

Proof of Proposition 2.

Proof. Following Duffie and Epstein (1992), define the (equilibrium) state price density π by

$$\pi(t) = \exp\left(\int_0^t \widetilde{f}_2\left(y(s), J\left(y(s), s\right), s\right) ds\right) \widetilde{f}_1\left(y(t), J\left(y(t), t\right), t\right),$$

then the equilibrium price at date t is given by

$$p(y(t)) = \frac{1}{\pi(t)} E_t\left(\int_t^\infty \pi(s) dD(s)\right)$$

where $D(s)$ is the cumulative dividend up to time s. In our case, we have assumed the cumulative dividend to be absolutely continuous and more precisely, $dD(s) = y(s)ds$. Easy computations lead to

$$\begin{aligned}
\widetilde{f}_1(c, v, t) &= \beta c^{\rho-1}(\alpha v)^{1-\frac{\rho}{\alpha}} e^{-\frac{\gamma\rho}{\alpha}t} \\
\widetilde{f}_2(c, v, t) &= \frac{\beta}{\rho}(\alpha - \rho)c^{\rho}(\alpha v)^{-\frac{\rho}{\alpha}} e^{-\frac{\gamma\rho}{\alpha}t} - (\frac{\alpha\beta}{\rho} - \gamma)
\end{aligned}$$

Since $J(y, t) = Ke^{-\gamma t}y^{\alpha}$, it follows that

$$\begin{aligned}
\widetilde{f}_1(y, J(y, t), t) &= \beta(\alpha K)^{1-\frac{\rho}{\alpha}} y^{\alpha-1} e^{-\gamma t} \\
\widetilde{f}_2(y, J(y, t), t) &= \frac{\beta}{\rho}(\alpha - \rho)(\alpha K)^{-\frac{\rho}{\alpha}} - (\frac{\alpha\beta}{\rho} - \gamma)
\end{aligned}$$

Using the expression obtained for K in appendix 1, after some straightforward manipulations, we obtain

$$\widetilde{f}_2(y, J(y, t), t) = -\alpha\left(\mu + \frac{(\alpha - 1)}{2}\sigma^2\right) - \left(\beta + \rho\left(-\mu + \frac{(1 - \alpha)}{2}\sigma^2\right)\right) + \gamma$$

To compute the equilibrium price, it is worth noting that $\overline{f}_2$ is a constant, so by the definition of the state price density π, we have the following relationship

$$\frac{\pi(s)}{\pi(0)} = e^{(\widetilde{f}_2-\gamma)s} \frac{y(s)^{\alpha-1}}{y^{\alpha-1}}.$$

Now define a new process X such that $X(s) = e^{(\widetilde{f}_2-\gamma)s} y(s)^{\alpha}$. Using Ito's lemma, we obtain

$$dX(s) = X(s)\left((\widetilde{f}_2 - \gamma + \alpha\mu + \frac{\alpha(\alpha - 1)}{2}\sigma^2)ds + \alpha\sigma dw(s)\right)$$

Taking expectation on both sides yields

$$\begin{aligned}
d(E_0(X(s))) &= (\widetilde{f}_2 - \gamma + \alpha\mu + \frac{\alpha(\alpha-1)}{2}\sigma^2)E_0(X(s))\,ds \\
&= -\left(\beta + \rho(\mu + \frac{(1-\alpha)}{2}\sigma^2)\right)E_0(X(s))\,ds
\end{aligned}$$

Since $X(0) = y^\alpha$, we obtain that

$$E_0(X(s)) = y^\alpha e^{-\left(\beta+\rho(-\mu+\frac{(1-\alpha)}{2}\sigma^2)\right)s}$$

Of course, the previous quantity is independent of γ. Thus, the equilibrium price at date 0 is given by

$$\begin{aligned}
p(y) &= \frac{1}{y^{\alpha-1}}\int_0^\infty E_0(X(s))\,ds \\
&= \frac{y}{\beta + \rho\left(-\mu + \frac{(1-\alpha)}{2}\sigma^2\right)}
\end{aligned}$$

Since $s = \frac{1}{1-\rho}$ or equivalently $\rho = \frac{s-1}{s}$ we obtain the desired result. ■

5.3. APPENDIX 3

Proof of Proposition 3.

Proof. Given a wealth distribution $Z = (z_1, z_2, ..., z_N)$ and a level of dividend y, denoting $V_i^*(y,Z)$ the **equilibrium** value function for agent i, the HBJ equation associated with agent i program is

$$\begin{aligned}
0 = {} & \max_{c_i} \frac{\beta_i}{\rho_i}\frac{c_i^{\rho_i} - (\alpha_i V_i^*(y,Z))^{\frac{\rho_i}{\alpha_i}}}{(\alpha_i V_i^*(y,Z))^{\frac{\rho_i}{\alpha_i}-1}} + \\
& \frac{1}{p}\frac{\partial V_i^*(y,Z)}{\partial z_i}[z_i y - c_i] + \sum_{k\neq i}\frac{1}{p}\frac{\partial V_i^*(y,Z)}{\partial z_k}[z_k y - c_k] \\
& +\mu y\frac{\partial V_i^*(y,Z)}{\partial y} + \frac{\sigma^2}{2}y^2\frac{\partial^2 V_i^*(y,Z)}{\partial y^2}
\end{aligned}$$

The first order condition is:

$$\beta_i\frac{(c_i^*)^{\rho_i-1}}{(\alpha_i V_i^*(y,Z))^{\frac{\rho_i}{\alpha_i}-1}} = \frac{1}{p}\frac{\partial V_i^*(y,Z)}{\partial z_i}$$

Plugging back in to the HBJ, we obtain

$$
\begin{aligned}
0 &= \frac{1-\rho_i}{\rho_i}\left(\frac{(\alpha_i V_i^*(y,Z))^{\frac{\rho_i}{\alpha_i}-1}}{\beta_i}\right)^{\frac{1}{\rho_i-1}}\left(\frac{1}{p}\frac{\partial V_i^*(y,Z)}{\partial z_i}\right)^{\frac{\rho_i}{\rho_i-1}}+ \\
&\quad \frac{1}{p}\frac{\partial V_i^*(y,Z)}{\partial z_i}z_i y-\alpha_i V_i^*(y,Z)+\sum_{k\neq i}\frac{1}{p}\frac{\partial V_i^*(y,Z)}{\partial z_k}\left[z_k y-c_k\right] \\
&\quad +\mu y\frac{\partial V_i^*(y,Z)}{\partial y}+\frac{\sigma^2}{2}y^2\frac{\partial^2 V_i^*(y,Z)}{\partial y^2}
\end{aligned}
$$

We now need to check that indeed $V_i^*(y,Z)=\Psi_i(Z)y^{\alpha_i}$, $c_i^*(y,Z)=C_i(Z)y$, $p(y,Z)=P(Z)y$, Z deterministic, and can be an equilibrium.

Plugging back $V_i^*(y,Z)=\Psi_i(Z)y^{\alpha_i}$ into the HBJ equation yields

$$
\begin{aligned}
0 &= \frac{1-\rho_i}{\rho_i}\left(\frac{(\alpha_i\Psi_i(Z))y^{\rho_i-\alpha_i}}{\beta_i}\right)^{\frac{1}{\rho_i-1}}\left(\frac{1}{P(Z)y}\frac{\partial\Psi_i(Z)}{\partial z_i}y^{\alpha_i}\right)^{\frac{\rho_i}{\rho_i-1}} \\
&\quad +\frac{1}{P(Z)y}\frac{\partial\Psi_i(Z)}{\partial z_i}y^{\alpha_i+1}z_i-\alpha_i\Psi_i(Z)y^{\alpha_i} \\
&\quad +\sum_{k\neq i}\frac{1}{P(Z)y}\frac{\partial\Psi_i(y,Z)}{\partial z_k}y^{\alpha_i}\left[z_k y-C_k(Z)y\right]+\alpha_i\mu\Psi_i(Z)y^{\alpha_i}+\frac{\sigma^2}{2}\alpha_i(\alpha_1-1)\Psi_i(Z)y^{\alpha_i}
\end{aligned}
$$

The expression is homogenous of degree α_i in y. This leads to

$$
\begin{aligned}
0 &= \frac{1-\rho_i}{\rho_i}\left(\frac{(\alpha_i\Psi_i(Z))}{\beta_i}\right)^{\frac{1}{\rho_i-1}}\left(\frac{1}{P(Z)}\frac{\partial\Psi_i(Z)}{\partial z_i}\right)^{\frac{\rho_i}{\rho_i-1}} \\
&\quad +\frac{1}{P(Z)}\frac{\partial\Psi_i(Z)}{\partial z_i}z_i-\alpha_i\Psi_i(Z) \\
&\quad +\sum_{k\neq i}\frac{1}{P(Z)}\frac{\partial\Psi_i(Z)}{\partial z_k}\left[z_k-C_k(Z)\right]+\alpha_i\mu\Psi_i(Z)+\frac{\sigma^2}{2}\alpha_i(\alpha_i-1)\Psi_i(Z)
\end{aligned}
$$

Indeed, assume Ψ_i satisfies an ODE. Writing $\Psi_i(Z(t))=J_i(t)$ for all t, we show in appendix 4 that J_i satisfies a Bernoulli ODE type and the problem becomes equivalent to solving a usual (deterministic) program when agents have CRRA utility functions for a Lucas tree economy. Therefore, there is a unique solution to the ODE satisfied by Ψ_i.

Moreover from the FOC, we obtain

$$
\beta_i(c_i^*)^{\rho_i-1} = (\alpha_i\Psi_i(Z)y^{\alpha_i})^{\frac{\rho_i}{\alpha_i}-1}\frac{1}{P(Z)}\frac{\partial\Psi_i(Z)}{\partial z_i}y^{\alpha_i-1}
$$

$$= \quad (\alpha_i \Psi_i(Z))^{\frac{\rho_i}{\alpha_i}-1} \frac{1}{P(Z)} \frac{\partial \Psi_i(Z)}{\partial z_i} y^{\rho_i - 1}$$

It is then easy to see that the optimal consumption can be written

$$c_i^*(y, Z) = C_i(Z)y.$$

From the budget constraint of agent i we obtain

$$\begin{aligned}
dz_i(t) &= \frac{1}{p(t)} \left[z_i(t)y(t) - c_i^*(t) \right] dt \\
&= \frac{1}{P(t)y(t)} \left[z_i(t)y(t) - C_i(Z(t))y(t) \right] dt \\
&= \frac{1}{P(t)} \left[z_i(t) - C_i(Z(t)) \right] dt
\end{aligned}$$

Hence

$$z_i(t) = e^{-\int_0^t \frac{ds}{P(s)}} \left(z_{i0} - \int_0^t e^{\int_0^s \frac{ds}{P(s)}} \frac{C_i(Z(s))ds}{P(s)} \right).$$

So indeed, the assumption Z is a deterministic function of time is compatible with the expression of z_i for $i = 1, 2, ..., N$.

The last thing we need to check is that we can indeed write $V_i^*(y, Z) = \Psi_i(Z)y^{\alpha_i}$. From the expression of the equilibrium value function we get

$$\begin{aligned}
V_i^*(y, Z) &= E_0 \left[\int_0^\infty \frac{\beta_i}{\rho_i} \left(\frac{(C_i(Z(s))y(s))^{\rho_i}}{(\alpha_i \Psi_i(Z(s))y(s)^{\alpha_i})^{\frac{\rho_i}{\alpha_i}-1}} - \alpha_i \Psi_i(Z(s))y(s)^{\alpha_i} \right) ds \right] \\
&= E_0 \left[\int_0^\infty \frac{\beta_i}{\rho_i} \left(\frac{(C_i(Z(s)))^{\rho_i}}{(\alpha_i \Psi_i(Z(s)))^{\frac{\rho_i}{\alpha_i}-1}} - \alpha_i \Psi_i(Z(s)) \right) y(s)^{\alpha_i} ds \right] \\
&= \int_0^\infty \frac{\beta_i}{\rho_i} \left(\frac{(C_i(Z(s)))^{\rho_i}}{(\alpha_i \Psi_i(Z(s)))^{\frac{\rho_i}{\alpha_i}-1}} - \alpha_i \Psi_i(Z(s)) \right) E_0 \left[y(s)^{\alpha_i} \right] ds \text{ since } Z(s) \text{ is deterministic} \\
&= y^{\alpha_i} \int_0^\infty \frac{\beta_i}{\rho_i} \left(\frac{(C_i(Z(s)))^{\rho_i}}{(\alpha_i \Psi_i(Z(s)))^{\frac{\rho_i}{\alpha_i}-1}} - \alpha_i \Psi_i(Z(s)) \right) e^{\alpha_i(\mu - \frac{1-\alpha_i}{2}\sigma^2)s} ds
\end{aligned}$$

The desired result follows: indeed, $V_i^*(y, Z)$ can be written $\Psi_i(Z)y^{\alpha_i}$. ■

5.4. APPENDIX 4

Proof of Proposition 4 and Corollary 1.

Proof. Since Z is a deterministic function of time, setting $y = y(t)$, we can rewrite the equilibrium value function of agent i, $V_i^*(y, Z(t)) = \Psi_i(Z(t))y^{\alpha_i} = J_i(t)y^{\alpha_i}$ and $c_i^*(y, Z(t)) = C_i(Z(t))y = h_i(t)y$ where J_i and h_i are deterministic functions of time. Hence

$$\begin{aligned}
J_i(t)y^{\alpha_i} &= E_t\left[\int_t^\infty \frac{\beta_i}{\rho_i}\left(\frac{(h_i(s))y(s))^{\rho_i}}{(\alpha_i J_i(s))y(s)^{\alpha_i})^{\frac{\rho_i}{\alpha_i}-1}} - \alpha_i J_i(s)y(s)^{\alpha_i}\right)ds\right] \\
J_i(t)y^{\alpha_i} &= \int_t^\infty \frac{\beta_i}{\rho_i}\left(\frac{h_i(s)^{\rho_i}}{(\alpha_i J_i(s))^{\frac{\rho_i}{\alpha_i}-1}} - \alpha_i J_i(s)\right)E_t\left[y(s)^{\alpha_i}\right]ds
\end{aligned}$$

This leads to

$$J_i(t)e^{a_i t} = \int_t^\infty \frac{\beta_i}{\rho_i}\left(\frac{h_i(s)^{\rho_i}}{(\alpha_i J_i(s))^{\frac{\rho_i}{\alpha_i}-1}} - \alpha_i J_i(s)\right)e^{a_i s}ds$$

where $a_i = \alpha_i(\mu - (1-\alpha_i)\frac{\sigma^2}{2})$.

Differentiating with respect to t yields

$$\dot{J_i}\,(t) = -\frac{\beta_i}{\rho_i}\frac{h_i(t)^{\rho_i}}{(\alpha_i J_i(t))^{\frac{\rho_i}{\alpha_i}-1}} + \left(\frac{\alpha_i\beta_i}{\rho_i} - a_i\right)J_i(t)$$

J_i satisfies a Bernoulli's type ODE. Define $\Phi_i \equiv J_i^{\rho_i/\alpha_i}$. It is easy to verify that Φ_i satisfies the following linear ODE

$$\dot{\Phi_i}\,(t) = -\beta_i(\alpha_i)^{\frac{\rho_i}{\alpha_i}}h_i(t)^{\rho_i} + \theta_i\Phi_i(t)$$

This exactly means that our problem is equivalent to the one where agent i maximizes the usual separable utility

$$\begin{aligned}
\Phi_i(t) &= \int_t^\infty \beta_i(\alpha_i)^{\frac{\rho_i}{\alpha_i}}h_i^{\rho_i}(s)e^{-\theta_i(s-t)}ds \\
\text{s.t. } dz_i(s) &= \frac{1}{P(s)}\left[z_i(s) - h_i(s)\right]ds
\end{aligned}$$

The Hamiltonian of this program is

$$H_i(h_i, \lambda_i, z_i, t) = \beta_i(\alpha_i)^{\frac{\rho_i}{\alpha_i}}h_i^{\rho_i}e^{-\theta_i t} + \frac{\lambda_i}{P}\left[z_i - h_i\right]$$

The optimal conditions are

$$\begin{aligned}\frac{\partial H_i}{\partial h_i} &= 0 \\ \dot{\lambda}_i + \frac{\partial H_i}{\partial z_i} &= 0\end{aligned}$$

or equivalently

$$\begin{aligned}\rho_i \beta_i (\alpha_i)^{\frac{\rho_i}{\alpha_i}} h_i^{\rho_i - 1} e^{-\theta_i t} &= \frac{\lambda_i}{P} \\ \dot{\lambda}_i + \frac{\lambda_i}{P} &= 0\end{aligned}$$

Define the pseudo riskless interest rate for a unit dividend

$$r = \frac{\dot{P} + 1}{P}$$

and the optimal condition for agent i is

$$\frac{\dot{h}_i}{h_i} = \frac{r - \theta_i}{b_i} \tag{5.1}$$

where $b_i = 1 - \rho_i$. Moreover, the transversality condition is

$$\lim_{t \to \infty} z_i(t) \lambda_i(t) = 0 \ \blacksquare$$

From the optimal condition (5.1) for any couple (i, j) we have

$$b_i \frac{\dot{h}_i}{h_i} - b_j \frac{\dot{h}_j}{h_j} = \theta_j - \theta_i \tag{5.2}$$

Lemma 1. $\lim\limits_{t \to \infty} h_1(t) = 1$, $\lim\limits_{t \to \infty} h_i(t) = 0$, $i = 2, ..., N$, $\lim\limits_{t \to \infty} P(t) = \frac{1}{\theta_1}$ *and* $\lambda_i(t) \underset{+\infty}{\sim} K_i e^{-\theta_1 t}$, $K_i > 0$.

Proof. Integrating relationship (5.2) for agent 1 and agent $j \neq 1$, we obtain

$$b_1 \ln \frac{h_1(t)}{h_{10}} - b_j \ln \frac{h_j(t)}{h_{j0}} = (\theta_j - \theta_1) t \tag{5.3}$$

The RHS of the equality goes to $+\infty$ when t goes to $+\infty$. Since $h_1(t)$ and $h_j(t)$ are between 0 and 1, it must be the case that $\lim\limits_{t\to\infty} h_i(t) = 0$, $i = 2, ..., N$. Moreover, since $\sum\limits_{k=1}^{N} h_k = 1$, we obtain $\lim\limits_{t\to\infty} h_1(t) = 1$. Moreover, from (5.3), we have

$$h_j(t) = h_{j0}\left(\frac{h_1(t)}{h_{10}}\right)^{\frac{b_1}{b_j}} e^{-\frac{(\theta_j-\theta_1)}{b_j}t}$$

Since $\lim\limits_{t\to\infty} h_1(t) = 1$, this implies that

$$h_j(t) \underset{+\infty}{\sim} A_j e^{-\frac{(\theta_j-\theta_1)}{b_j}t},\ A_j > 0,\ j = 2, ..., N.$$

From $\sum\limits_{k=1}^{N} h_k = 1$, we deduce that

$$h_1(t) \underset{+\infty}{\sim} 1 - A_n e^{-\frac{(\theta_n-\theta_1)}{b_n}t}$$

with $n = \underset{2\leqslant j\leqslant N}{\arg\min} \frac{(\theta_j-\theta_1)}{b_j}$.

From the FOC, we obtain $\frac{\lambda_i(t)}{P(t)} \underset{+\infty}{\sim} B_i e^{-\theta_1 t}$, $B_i > 0$. In addition, it is easy to see that $\frac{\dot{P}(t)+1}{P(t)} \underset{+\infty}{\sim} \theta_1$. This implies $P(t) \underset{+\infty}{\sim} \frac{1}{\theta_1} + Ae^{\theta_1 t}$. Now assume that $A \neq 0$, we obtain $\lambda_i(t) \underset{+\infty}{\sim} B_i A$. This leads to a contradiction since we have $\lim\limits_{t\to\infty} h_1(t) = 1$ and we must have also $\lim\limits_{t\to\infty} h_1(t)\lambda_1(t) = 0$. Thus, $A = 0$, $\lim\limits_{t\to\infty} P(t) = \frac{1}{\theta_1}$ and $\lambda_i(t) \underset{+\infty}{\sim} K_i e^{-\theta_1 t}$, $K_i > 0$. ■

$\dot{h}_1 > 0$ **and** $\dot{h}_N < 0$

Proof. Since the marginal utility is infinite at 0, an agent always chooses to consume a positive amount so $h_i > 0$. Differentiating with respect to time the equilibrium condition $\sum\limits_{k=1}^{N} h_k = 1$, we obtain $\sum\limits_{k=1}^{N} \dot{h}_k = 0$. Moreover, $\dot{h}_i = h_i \frac{r-\theta_i}{b_i}$ and thus, $\left(\sum\limits_{k=1}^{N} \frac{h_k}{b_k}\right) r = \left(\sum\limits_{k=1}^{N} \frac{h_k}{b_k}\theta_k\right)$. Since $\theta_1 < \theta_k < \theta_N$ it is easy to see that $\theta_1 < r < \theta_N$, and consequently, $\dot{h}_1 > 0$ and $\dot{h}_N < 0$. ■

$h_1 < z_1$ **and** $h_N > z_N$

Proof. From agent i budget equation $dz_i(t) = \frac{1}{P(t)}\left[z_i(t) - h_i(t)\right]dt$ and the relationship $\frac{\dot{\lambda}_i}{\lambda_i} = -\frac{1}{P(t)}$, we have

$$d\left(z_i(t)\lambda_i(t)\right) = \dot{\lambda}_i(t)h_i(t)$$

Integrating between dates t and s $(t < s)$, we obtain

$$\begin{aligned} z_i(s)\lambda_i(s) - z_i(t)\lambda_i(t) &= \int_t^s \dot{\lambda}_i(u)h_i(u)du \\ &= \lambda_i(s)h_i(s) - \lambda_i(t)h_i(t) - \int_t^s \dot{h}_i(u)\lambda_i(u)du \end{aligned}$$

By the TVC, we have $\lim\limits_{s\to\infty} z_i(s)\lambda_i(s) = 0$. Moreover, since $\lim\limits_{s\to\infty} \lambda_i(s) = 0$, this implies that $\lim\limits_{s\to\infty} \lambda_i(s)h_i(s) = 0$ and it follows

$$\left(h_i(t) - z_i(t)\right)\lambda_i(t) = -\int_t^\infty \dot{h}_i(u)\lambda_i(u)du$$

We can conclude that for all t

$$\begin{aligned} h_1(t) &< z_1(t) \text{ since } \dot{h}_1(t) > 0 \\ h_N(t) &> z_N(t) \text{ since } \dot{h}_N(t) < 0 \blacksquare \end{aligned}$$

5.5. APPENDIX 5

Proof of Proposition 5

Proof. Recall that

$$r = \frac{\sum\limits_{k=1}^N \frac{h_k}{b_k}\theta_k}{\sum\limits_{k=1}^N \frac{h_k}{b_k}}. \tag{5.4}$$

Therefore since $\theta_1 < \theta_k < \theta_N$, we have $\theta_1 < r < \theta_N$. Moreover, we have seen that $\lim\limits_{t\to\infty} h_1(t) = 1$ and $\lim\limits_{t\to\infty} h_i(t) = 0$, $i = 2, ..., N$. This implies that $\lim\limits_{t\to\infty} r(t) = \theta_1 < \theta_2$. Therefore, for t large enough, we must have $r(t) < \theta_2$. Unless $r(0) < \theta_2$, choose $T = \min\limits_{t\geqslant 0} \{r(T) = \theta_2 \text{ and } \dot{r}(T) < 0\}$. We want to show that for all $t > T$, $r(t) < \theta_2$.

Lemma 2. *If* $h_i > 0$, $\sum\limits_{i=1}^N h_i = 1$, $\sum\limits_{i=1}^N \dot{h}_i = 0,$, $\dot{h}_1 > 0$ *and* $\dot{h}_i \leqslant 0$, $i = 2, ..., N$, *then* $\dot{r} < 0$.

Proof. Totally differentiating relationship (5.4) with respect to time yields

$$\begin{aligned}
\dot{r}\,(t) &= \frac{\left(\sum\limits_{k=1}^{N}\frac{\theta_k \dot{h}_k}{b_k}\right)\left(\sum\limits_{k=1}^{N}\frac{h_k}{b_k}\right)-\left(\sum\limits_{k=1}^{N}\frac{\theta_k h_k}{b_k}\right)\left(\sum\limits_{k=1}^{N}\frac{\dot{h}_k}{b_k}\right)}{D^2} \\
&= \frac{\left(\sum\limits_{1\leqslant i<j\leqslant N}^{N}(\theta_j-\theta_i)\left(\frac{\dot{h}_j}{b_j}\frac{h_i}{b_i}-\frac{\dot{h}_i}{b_i}\frac{h_j}{b_j}\right)\right)}{D^2}
\end{aligned}$$

where $D=\sum\limits_{k=1}^{N}\frac{h_k}{b_k}$. Set $\Psi=\left(\sum\limits_{1\leqslant i<j\leqslant N}^{N}(\theta_j-\theta_i)\left(\frac{\dot{h}_j}{b_j}\frac{h_i}{b_i}-\frac{\dot{h}_i}{b_i}\frac{h_j}{b_j}\right)\right)$. We want to show that

$$\begin{aligned}
&\max\Psi \\
\text{s.t. } h_i &\geqslant 0 \\
\sum_{i=1}^{N}h_i &= 1,\ \sum_{i=1}^{N}\dot{h}_i=0 \\
\dot{h}_1 &\geqslant 0 \text{ and } \dot{h}_i\leqslant 0,\ i=2,...,N.
\end{aligned}$$

is equal to zero. **Note that the constraints imposed on Ψ are less stringent than the ones for the desired lemma**. We examine the possible values of the $\left(\dot{h}_i,h_i\right)$ in order to reach the maximum value for Ψ.

Notice that $1\leqslant i<j\leqslant N$, $(\theta_j-\theta_i)\frac{\dot{h}_j}{b_j}\leqslant 0$, set $h_1=0$ unless each term $(\theta_j-\theta_i)\frac{\dot{h}_j}{b_j}$ is zero for $j=2,...,N$. The latter implies $\dot{h}_1=0$ and in this case, $\Psi=0$. If all the $\dot{h}_j$ are not equal to zero, we must choose $h_1=0$. By the same logic since $-(\theta_j-\theta_i)\frac{h_j}{b_j}\leqslant 0$, we must choose $\dot{h}_1=0$. Using a similar argument looking at the possible values for h_2 and $\dot{h}_2$ in order to maximize Ψ, once again we obtain that $h_2=0$ and $\dot{h}_2=0$. Then, we progressively eliminate each couple $\left(\dot{h}_i,h_i\right)$ for $i=3,...,N$ and we finally obtain that $\max\Psi=0$ exactly when $\left(\dot{h}_i,h_i\right)=(0,0)$ for all $i=1,...,N$. Note that at the equilibrium, $h_i>0$ for $i=1,2,...,N$ and $\dot{h}_1>0$, so it must be the case that Ψ is negative which implies $\dot{r}\,(t)<0$ as soon as $r(t)<\theta_2$. Note that $r(t)\leqslant\theta_2$ if and only if $\sum\limits_{k=1}^{N}\frac{h_k}{b_k}\theta_k\leqslant\theta_2\sum\limits_{k=1}^{N}\frac{h_k}{b_k}$ which implies

$$\sum_{k=3}^{N}\frac{h_k}{b_k}(\theta_k-\theta_2)\leqslant(\theta_2-\theta_1)\frac{h_1}{b_1}\ \blacksquare$$

Lemma 3. *For all date $t > T$, $r(t) < \theta_2$.*

Proof. For all $\delta > 0$ small enough $r(T+\delta) < \theta_2$, so $\dot{h}_i\ (T+\delta) < 0$, $i = 2, ..., N$ and therefore $\dot{r}\ (T+\delta) < 0$. Now assume that there exists a date $s > T$ such that $\dot{r}\ (s) = 0$ and $\dot{r}\ (t) < 0$ on $[T, s)$. Since we must have $r(s) < \theta_2$, this implies $\dot{h}_i\ (s) < 0$ for $i = 2, ..., N$, and consequently $\dot{r}\ (s) < 0$ which is a contradiction. Henceforth, $r(t) < \theta_2$ for all $t > T$. ■

Finally from lemma 1. and lemma 2., we conclude that for $t > T$, $r(t) < \theta_2$. The proof is complete ■

Equilibrium Price Properties

From lemma 2. and lemma 3., for all $t > T$, $\dot{r}\ (t) < 0$. Using relationship (3.2), we have

$$\frac{\dot{h}_i}{h_i} \text{ is strictly decreasing for } t > T,\ i = 1, 2, ..., N.$$

Recall that

$$\begin{aligned} P(t) &= \frac{\int_t^\infty h_i^{-b_i}(s) e^{-\theta_i(s-t)} ds}{h_i^{-b_i}(t)} \\ &= \int_0^\infty \left(\frac{h_i(u+t)}{h_i(t)}\right)^{-b_i} e^{-\theta_i u} du \end{aligned}$$

Thus

$$\dot{P}\ (t) = -b_i \int_0^\infty \left(\frac{h_i(u+t)}{h_i(t)}\right)^{-(b_i+1)} \frac{1}{h_i(t) h_i(u+t)} \left(\frac{\dot{h}_i}{h_i}(t+u) - \frac{\dot{h}_i}{h_i}(t)\right) e^{-\theta_i u} du$$

It follows that for $t > T$, $\dot{P}\ (t) > 0$, so P is increasing.

Moreover, for $t > T$,

$$\begin{aligned} \int_0^\infty (1)^{-b_2}\, e^{-\theta_2 u} du &< P(t) < \int_0^\infty (1)^{-b_1}\, e^{-\theta_1 u} du \\ \frac{1}{\theta_2} &< P(t) < \frac{1}{\theta_1}\ ■ \end{aligned}$$

Equilibrium Consumption Properties

We have already seen in appendix 4 that $\lim_{t\to\infty} h_1(t) = 1$, $\dot{h_1} > 0$, $\dot{h_N} > 0$ and $\lim_{t\to\infty} h_i(t) = 0$ for $i = 2, ..., N$. In addition, for $t > T$, $r(t) < \theta_2$. Thus for $i = 2, 3, ..., N-1$, we have $\dot{h_i}\ (t) < 0$ for $t > T$.

Portfolio Shares and Borrowers and Lenders

We have already seen in appendix 4 that $h_1 < z_1$, $h_N > z_N$ and for all $i = 1, 2, ..., N$

$$(h_i(t) - z_i(t))\,\lambda_i(t) = -\int_t^\infty \dot{h_i}\ (u)\lambda_i(u)du \tag{5.5}$$

Moreover, for $t > T$, $\dot{h_i}\ (t) < 0$. Therefore, for $t > T$, $h_i(t) > z_i(t)$. Finally from agent i's budget constraint $dz_i(t) = \frac{1}{P(t)}\left[z_i(t) - h_i(t)\right]dt$, we conclude that $\dot{z_1} > 0$, $\dot{z_N} < 0$ since , $h_N > z_N$ and for $t > T$, $\dot{z_i}\ (t) < 0$ since $h_i(t) > z_i(t)$. Finally, since for $i = 2, 3, ..., N$ we have $h_i(t) \underset{+\infty}{\sim} A_i e^{-\frac{(\theta_i - \theta_1)}{b_i}t}$, it follows that $\dot{h_i}\ (t) \underset{+\infty}{\sim} -A_i \frac{(\theta_i-\theta_1)}{b_i} e^{-\frac{(\theta_i-\theta_1)}{b_i}t}$. Moreover, $\lambda_i(t) \underset{+\infty}{\sim} K_i e^{-\theta_1 t}$. From relationship (5.5), we obtain

$$(h_i(t) - z_i(t))\,\lambda_i(t) \underset{+\infty}{\sim} A_i K_i \frac{(\theta_i-\theta_1)}{b_i} \int_t^\infty e^{-\left[\frac{(\theta_i-\theta_1)}{b_i}+\theta_1\right]u} du \text{ so}$$
$$A_i e^{-\frac{(\theta_i-\theta_1)}{b_i}t} - z_i(t) \underset{+\infty_i}{\sim} \frac{A_i \frac{(\theta_i-\theta_1)}{b_i}}{\frac{(\theta_i-\theta_1)}{b_i}+\theta_1} e^{-\frac{(\theta_i-\theta_1)}{b_i}t} \text{ with } E_i > 0$$

It follows

$$z_i(t) \underset{+\infty}{\sim} F_i e^{-\frac{(\theta_i-\theta_1)}{b_i}t}, \text{ with } F_i > 0,\ i = 2, 3, ..., N.$$

This implies $\lim_{t\to\infty} z_i(t) = 0$ for $i = 2, 3, ..., N$ and since $\sum_{i=1}^{N} z_i = 1$, $\lim_{t\to\infty} z_1(t) = 1$.

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