Opting Out in a War of Attrition

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Abstract

This paper analyzes the role played by the outside options in negotiations when there is incomplete information about their existence. We examine a War of Attrition where players enjoy private information about their possibility of leaving the negotiation to take an outside option. The main message that emerges from the analysis of this game is that uncertainty about the possibility that the opponent opts out improves efficiency, since it increases the equilibrium probability of concession. More precisely, if the probability that the opponent is strong is relatively high, in equilibrium, the negotiation eventually ends with a sure concession. On the other extreme, if the likelihood of a weak opponent is high, strong types will eventually leave the negotiation and opt out with probability 1 leaving weak types to play from that time on the inefficient symmetric equilibrium of the classical War of Attrition. Even in this case, the probability of concession along the uncertainty phase of the equilibrium play increases.

Keywords: war of attrition, outside options.

1 Introduction

This aim of this paper is to study the role played by the outside options in negotiations when there is incomplete information about their existence. For this purpose we focus our analysis on the War of Attrition since this is the simplest model of conflict that yields inefficient equilibria under complete information. It is well known that, in a symmetric War of Attrition without outside options, the unique symmetric equilibrium consists in players randomizing at a constant probability between conceding and not conceding, a very inefficient outcome indeed. We show that the presence of uncertain outside options improves efficiency.

The relevance of outside opportunities available to the players on the outcome of a negotiation has been well established in models of bargaining with complete information (Shaked and Sutton (1984), Binmore et al.(1986), Shaked (1987), and Ponsati and Sakovics (1998)). In these models the decision of a bargainer to take up her outside option is a strategic decision and outcomes depend crucially on who has this possibility and when. If it is the responder who has the outside opportunity, then, in the unique subgame perfect equilibrium, this player obtains a payoff equal to the value of her option if this is larger than her equilibrium share in the game without the possibility to opt out. Otherwise, the option has no effect on the outcome (this is known as Outside Option Principle, see Shaked and Sutton (1984)). But if it is the proposer who can threaten to take her outside option, she can appropriate the entire surplus making a take-it-or-leave-it offer and, in this case, there is multiplicity of equilibria for a range of outside options.

Considering uncertainty about outside options is a natural extension of the literature that deserves attention. Nevertheless, bargaining models devoted to that subject are scarce.[1] Wolinsky (1987) presents a model

[1] The literature of pretrial negotiation has an apparent similarity to models of sequential bargaining with incomplete information and outside options. In these

where players may search for outside opportunities during the bargaining process. He shows that the outcome of the bargaining does not depend only on the players' relative efficiency in searching, but also on how aggressively each party can credibly threaten to search in the event that the agreement is delayed. Vislie (1988) extends Shaked and Sutton's model (1984) by allowing the presence of a second random outside option for the seller, and finds the conditions under which the equilibrium price is affected by this random appearance. And finally, Ponsati and Sakovics (1999) analyze a bargaining game where both players have outside options but they are uncertain about their size. In all these models players do not know with certainty either the existence or the size of their own outside options. By contrast, in this paper we present a model where players enjoy private information about their possibilities of opting out, but they do not know their opponent' opportunities.

We carry out our analysis within the simple framework of a War of Attrition, a situation where there are only two available agreements and each player favors one of them. The decision problem of each player consists in deciding when to give in by accepting her opponent´s favorite agreement. The distinctive feature of our model is that, since outside options are present yielding takes two forms: a player can give in by accepting her opponent's favorite agreement, or by contrast, she can give up, taking her option, and leaving her current partner to take her outside payoff as well. Both players have private information about their own outside options and are impatient in that delaying is costly. There are two types of players: a weak type who has no outside option (or whose outside option is without value) and a strong type who has a

models of litigation and pretrial negotiation (see Spier (1992), Wang et al, (1994)) only the plaintiff can opt out forcing the trial. But an important difference is that in bargaining models both players would like to come to an agreement immediately while in pretrial negotiation the plaintiff would like to settle as soon as possible and the defendant to pay as late as possible.

valuable outside option that she prefers to take rather than conceding.

We show that introducing the possibility of opting out in a War of Attrition has a dramatic effect on the outcomes.[2] We find that, if the probability of facing a weak opponent is sufficiently low, in equilibrium, the negotiation will surely end at some future date, since weak types eventually become sufficiently pessimistic about the prospect of reaching their preferred agreement so that, in fear that the opponent might opt out, they concede with probability 1. On the other extreme, if the likelihood of a weak opponent is high, strong types eventually opt out with probability 1, leaving weak types to play, from that time on, the symmetric inefficient equilibrium of the complete information War of Attrition. Even in this case, the probability of concession along the uncertainty phase of the equilibrium play increases.

The following section presents our bargaining model and characterizes equilibria of this game. In section 3 we turn to an asymptotic analysis of this game considering the limit as $\delta \to 1$, and carry out comparative statics. Conclusions are presented in the last section.

[2] In a very different model, Compte and Jehiel (2000) find also that outside options have a positive effect on bargaining. They show that the existence of outside options may cancel out the effect of obstinacy in bargaining.

2 The model

The following bargaining situation is studied. Two players bargain about how to share one unit of surplus that will be available only when they reach an agreement. An agreement is denoted by x, where x indicates the portion of the surplus assigned to player 1. There are only two possible agreements; either $x = 1 - a$ or $x = a$ with $0 < a < \frac{1}{2}$. Players may also decide to break the negotiation by opting out, in which case, they receive a payoff b_i $i = 1, 2$.

In this game there are three possible bargaining outcomes; either an agreement is reached, or negotiations break, or perpetual disagreement prevails.

Players are assumed to be risk neutral and impatient. Their impatience is modeled by a common discount factor, normalized to be δ per unit of time. And the payoffs are as follows: if players perpetually disagree, they both receive zero payoff. If only player i concedes at time t, then player i gets $a\delta^t$ and player j gets $(1-a)\delta^t$. If both players concede at the same time each players gets $a\delta^t$[3]. And if either or both players opt out, payoffs are $b_i\delta^t$ for $i = 1, 2$.

Each player i has private information about the value of her outside opportunity, which can be either $b_i = 0$ or $b_i = b$, $a < b < 1 - a$. A player with no outside option (or whose outside option is 0) is a weak type, denoted as W, and a player with an outside option $b > 0$ is a strong type, denoted as S. Strong types always prefer opting out rather than conceding and weak types prefer conceding rather than opting out. The players entertain beliefs about each other's type and they are represented by an initial probability $0 < \pi_0^i < 1$, that is, the probability that player i is weak. We assume that these probabilities are common knowledge

[3]This assumption is computationally convenient. Results do not change substantially if we assume that in the case that both players concede at the same time a lottery is used to decide the outcome.

and we set $\pi_0^i = \pi_0$ for simplicity.

The game is played in discrete time, starting at $t = 0$. At each time (a stage), both players decide simultaneously either: (i) to propose her preferred agreement, (ii) to concede by proposing her opponent's favorite agreement or (iii) to leave the negotiation and opt out. The game ends whenever a player or both, at the same time, concedes or opts out. Otherwise, disagreement occurs, discounting applies and the game proceeds to a new stage.

The history h_t observed by the players is just the fact that no player has yielded before t (no player has conceded or has opted out).

A strategy $\sigma_i(\tau)$ of player i with type $\tau = W, S$ is defined as a pair of sequences $\sigma_i(\tau) = \left\{\alpha_t^i(\tau), \beta_t^i(\tau)\right\}_{t=0}^{\infty}$ where $\alpha_t^i(\tau)$ is the probability of conceding at t and $\beta_t^i(\tau)$ is the probability of opting out at t, given that no player yields before that time. Let $\sigma = (\sigma_i(W), \sigma_i(S), \sigma_j(W), \sigma_j(S))$.

A system of beliefs π^i for player i maps each observed history into some probability measure on the types W and S of player j. Let $\Pi = (\pi^i, \pi^j)$.

Given a strategy-belief profile (σ, Π), the expected payoff of player i of not conceding at t, conditional on the history h_t, is

$$V_t^{iW} = \pi_t^j \alpha_t^j (1-a) + \delta \left[1 - \pi_t^j \alpha_t^j - (1-\pi_t^j)\beta_t^j\right] V_{t+1}^{iW},$$

and the expected payoff of not opting out at t is

$$V_t^{iS} = \pi_t^j \alpha_t^j (1-a) + (1-\pi_t^j)\beta_t^j b_i + \delta \left[1 - \pi_t^j \alpha_t^j - (1-\pi_t^j)\beta_t^j\right] V_{t+1}^{iS}.$$

Since we are interested on the role played by outside options on the efficiency and outcome of the War of Attrition, we find appropiate to examine the Symmetric Perfect Bayesian Equilibria of this game given that inefficiency arises in a War of Attrition when players are constrained to use symmetric strategies.

The *Symmetric Perfect Bayesian Equilibrium* (SPBE) is defined in the usual way. A strategy-belief profile (σ, Π) is a SPBE if, at any stage of the game, strategies are optimal given the beliefs, and the beliefs are obtained from equilibrium strategies and observed actions using Bayes' rule:

$$\pi_t^i = \frac{\pi_{t-1}^i(1-\alpha_{t-1}^i)}{\pi_{t-1}(1-\alpha_{t-1}^i)+(1-\pi_{t-1}^i)(1-\beta_{t-1}^i)}.$$

Notice that π_t^i is not defined if $\alpha_{t-1}^i = \beta_{t-1}^i = 1$. If the optimal strategy tells a player to concede and opt out at some t with probability 1, then to stay at $t+1$ is a probability 0 event and Bayes' rule does not pin down posterior beliefs. Any posterior beliefs are then admissible. Symmetry in strategies implies that $\alpha_t^i = \alpha_t^j = \alpha_t$ and $\beta_t^i = \beta_t^j = \beta_t$.

Since in a SPBE a weak type will never opt out and a tough type will never concede, in an abuse of terminology, we will identify the probabilities of conceding α_t with the strategy of the weak type, and the probabilities of opting out β_t with the strategy of the tough type.

The first result is quite straight forward.

Proposition 1. *There is no SPBE in pure strategies.*

Proof. See Appendix.■

We next turn attention to profiles where players randomize. In a SPBE in mixed strategies, it must be true that the payoff of conceding at t, conditional on the opponent not having conceded or opted out previously, must be equal to the payoff of conceding at t+1. At the same time, the payoff of opting out at t, conditional on the opponent not having yielded in before, must be equal to the payoff of opting out at t+1:

$$\begin{gathered} a = (1-a)\pi_t\alpha_t + a\delta(1-\pi_t\alpha_t-(1-\pi_t)\beta_t), \\ b = (1-a)\pi_t\alpha_t + b(1-\pi_t)\beta_t + b\delta(1-\pi_t\alpha_t-(1-\pi_t)\beta_t). \end{gathered} \tag{1}$$

Next lemma points out that, in a SPBE it is not possible to have both types yielding at the same time with probability 1. And if the equilibrium is such that weak types concede with probability 1 at some t, then strong types certainly opt out at t+1.

Lemma 1. *If* $\{\alpha_t\}_0^\infty$ *and* $\{\beta_t\}_0^\infty$ *are SPBE, then:*
(i) there is no t *such that* $\alpha_t = \beta_t = 1$
(ii) If $\alpha_t = 1$ *and* $0 < \beta_t < 1$ *then* $\beta_{t+1} = 1$.

Proof. Statement (i) indicates that, in equilibrium, it is not possible that both types yield at the same time with probability 1. If the strategy of the opponent is to concede and to opt out at some t with probability 1, then a strong player will have always incentives to wait one period since $b < (1-a)\pi_t + b(1-\pi_t)$, breaking the symmetry of the strategies. Statement (ii) establishes that, if the weak type strategy yields a period t probability of conceding of unity, then to opt out at t+1 dominates doing so in t+2, since waiting until period t+2 discounts their payoff and provides no additional probability that a weak type will make a concession.■

In a SPBE, both types distribute concessions across time. The equilibrium strategies are characterized by the pair of difference equations (1). To simplify notation let,

$$H = \frac{ab(1-\delta)}{a\delta(1-a-\delta b) + b(1-\delta)(1-a-\delta a)},$$
$$G = \frac{(1-\delta)(1-a)(b-a)}{a\delta(1-a-\delta b) + b(1-\delta)(1-a-\delta a)}.$$

Our system of equations (1) can be rewritten as

$$\alpha_t \pi_t = H.$$
$$\beta_t(1-\pi_t) = G.$$

Substituting these expressions on the posterior probability π_t, we have the difference equation that rules the posterior:

$$\pi_t - \frac{\pi_{t-1}}{1-H-G} + \frac{H}{1-H-G} = 0.$$

Solving this difference equation with the initial condition π_0,

$$\pi_t = \frac{H}{H+G} + (\pi_0 - \frac{H}{H+G}) \left(\frac{1}{1-H-G} \right)^t.$$

For what follows we analyze the different profiles that can be sustained as equilibria.

Concession Equilibria

A *Concession Strategy Profile* is a strategy profile where weak types eventually concede with probability 1.

Define $\underline{T}$ as the natural number that solves:

$$\tfrac{H}{H+G} + (\pi_0 - \tfrac{H}{H+G})(\tfrac{1}{1-H-G})^t \leq H \leq \tfrac{H}{H+G} + (\pi_0 - \tfrac{H}{H+G})(\tfrac{1}{1-H-G})^{t-1}.$$

Our result, stated below as Proposition 2, shows that if the initial probability of facing a weak type is $\pi_0 \in \left(0, \frac{H}{H+G}\right)$ in equilibrium players will not continue in the game indefinitely. Instead, we can identify a period $\underline{T}$, which depends upon the parameters of the game (a, b, δ, π_0), with the property that weak types will never delay play beyond period $\underline{T}$ and strong types never stay beyond $\underline{T}+1$. Moreover, if $\pi_0 \in (0, H]$, the game ends at $\underline{T} = 0$.

A Concession Equilibrium is described by finite pairs of sequences $\{\alpha_t\}_{t=0}^{\underline{T}}$ and $\{\beta_t\}_{t=0}^{\underline{T}+1}$ identifying the indifference valuations in each period and a sequence of beliefs $\{\pi_t\}_{t=0}^{\underline{T}}$. The posterior π_t deteriorates over time; as time passes, players become more pessimistic about their opponents being a weak type. That fact will naturally affect the probability of conceding α_t which increases over time and the probability of opting out β_t which decreases. At some time $t = \underline{T}$ the probability that her opponent is strong is so high that a weak type optimally concedes with probability 1 since the chance to receive her preferred agreement is too small. And, as stablished on Lemma 1, a strong type will opt out at $\underline{T}+1$ with probability 1 if this agent infers that her opponent is strong.

If the period $\underline{T}$ is reached by which a weak type would have conceded, the strong type infers that the opponent is as strong as she is. If both players prefer opting out rather than conceding, they will leave the negotiation immediately since there is no possibility to receive $1-a$ from their opponents and delaying their way out only decreases their payoffs.

The formal statement of this result follows:

Proposition 2. *If $\pi_0 \in (0, H]$, there is a unique SPBE such that $\alpha_t = \beta_{t+1} = 1$ $\forall t \geq 0$ and $\beta_0 = \dfrac{b(1-\delta) - \pi_0(1-a-\delta b)}{b(1-\delta)(1-\pi_0)}$. And if $\pi_0 \in \left(H, \frac{H}{H+G}\right)$, the unique SPBE is such that:*

$$\alpha_t = \frac{H}{\frac{H}{H+G} + (\pi_0 - \frac{H}{H+G})\left(\frac{1}{1-H-G}\right)^t}, \ \forall t < \underline{T},$$

$$\beta_t = \frac{G}{1 - \left[\frac{H}{H+G} + (\pi_0 - \frac{H}{H+G})\left(\frac{1}{1-H-G}\right)^t\right]}, \ \forall t \leq \underline{T},$$

and $\alpha_t = \beta_{t+1} = 1$ $\forall t \geq \underline{T}$.

Proof. We prove Proposition 2 for $\pi_0 \leq H$. The rest is detailed in the appendix.

Let us check first the optimal response of both types to the opponent's strategy $(\{\alpha_t\}_0^\infty \{\beta_t\}_0^\infty)$ such that $\alpha_t = \beta_{t+1} = 1$ $\forall t \geq 0$ and $\beta_0 = \frac{b(1-\delta)-\pi_0(1-a-\delta b)}{b(1-\delta)(1-\pi_0)}$. Notice that, given this strategy of the opponent, $\pi_t = 0$ $\forall t \geq 1$.

A weak type concedes optimally at t=0 if:

$$a > \pi_0\alpha_0(1-a) + a\delta(1 - \pi_0\alpha_0 - (1-\pi_0)\beta_0), \text{ for } t = 0.$$

$$a > \pi_t\alpha_t(1-a) + a\delta(1 - \pi_t\alpha_t - (1-\pi_t)\beta_t), \ \forall t \geq 1.$$

The second inequality is automatically satisfied since $\pi_t = 0$. And the first inequality is satisfied since $\pi_0 \leqslant H$.

Consider now a strong type. If the strategy of her opponent is to concede at t=0 with probability 1, then, by Lemma 1, she will opt out with probability 1 at t=1. And at t=0 she opts out with probability $\beta_0 = \frac{1}{1-\pi_0} - (\frac{\pi_0}{1-\pi_0})\frac{(1-a-\delta b)}{b(1-\delta)}$ since $b = \pi_0(1-a) + b(1-\pi_0)\beta_0 + b\delta(1-\pi_0)(1-\beta_0)$.

Now we prove that if $\pi_0 \leq H$, then the unique SPBE must be $(\{\alpha_t\}_0^\infty, \{\beta_t\}_0^\infty)$ such that $\alpha_t = 1\ \forall t \geq 0$ and $\beta_0 = \frac{b(1-\delta)-\pi_0(1-a-\delta b)}{b(1-\delta)(1-\pi_0)}$ $\beta_t = 1\ \forall t \geq 1$. To see that, indeed this is the unique SPBE, we explore all the other possible candidates.

First, assume that there is a SPBE $(\left\{\tilde{\alpha}_t\right\}_0^\infty, \left\{\tilde{\beta}_t\right\}_0^\infty)$ with $0 < \tilde{\alpha}_0 < 1$, $0 < \tilde{\beta}_0 < 1$ and $\tilde{\beta}_0 \neq \frac{b(1-\delta)-\pi_0(1-a-\delta b)}{b(1-\delta)(1-\pi_0)}$. If these were equilibrium strategies, then it must be true that:

$$\pi_0\ \tilde{\alpha}_0 = H.$$
$$(1-\pi_0)\ \tilde{\beta}_0 = 1 - G.$$

But since $\pi_0 \leq H$ then $\tilde{\alpha}_0 \geq 1$, a contradiction.

Second, assume that $(\left\{\tilde{\alpha}_t\right\}_0^\infty, \left\{\tilde{\beta}_t\right\}_0^\infty)$ is an equilibrium with $\tilde{\alpha}_0 = 1$ and $0 < \tilde{\beta}_0 < 1$ with $\tilde{\beta}_0 \neq \frac{b(1-\delta)-\pi_0(1-a-\delta b)}{b(1-\delta)(1-\pi_0)}$. Then if these strategies constitute a SPBE, it must be true that:

$$a > (1-a)\pi_0 + a\delta(1-\pi_0)(1-\tilde{\beta}_0),$$

and

$$b = (1-a)\pi_0 + b(1-\pi_0)\ \tilde{\beta}_0 + b\delta(1-\pi_0)(1-\tilde{\beta}_0).$$

But if $\tilde{\beta}_0 \neq \frac{b(1-\delta)-\pi_0(1-a-\delta b)}{b(1-\delta)(1-\pi_0)}$, the second condition is violated; either a strong type will deviate by opting out at t=0 if $\tilde{\beta}_0 < \frac{b(1-\delta)-\pi_0(1-a-\delta b)}{b(1-\delta)(1-\pi_0)}$ or by never opting out if $\tilde{\beta}_0 > \frac{b(1-\delta)-\pi_0(1-a-\delta b)}{b(1-\delta)(1-\pi_0)}$.

Finally, assume that $(\left\{\tilde{\alpha}_t\right\}_0^\infty, \left\{\tilde{\beta}_t\right\}_0^\infty)$ is an equilibrium with $0 < \tilde{\alpha}_0 < 1$ and $\tilde{\beta}_0 = \frac{b(1-\delta)-\pi_0(1-a-\delta b)}{b(1-\delta)(1-\pi_0)}$. Then,

$$a = (1-a)\pi_0\ \tilde{\alpha}_0 + a\delta(1-\pi_0\ \tilde{\alpha}_0 - (1-\pi_0)\ \tilde{\beta}_0).$$

$$b = (1-a)\pi_0\ \tilde{\alpha}_0 + b(1-\pi_0)\ \tilde{\beta}_0 + b\delta(1-\pi_0\ \tilde{\alpha}_0 - (1-\pi_0)\ \tilde{\beta}_0).$$

But if $\widetilde{\beta}_0 = \frac{b(1-\delta)-\pi_0(1-a-\delta b)}{b(1-\delta)(1-\pi_0)}$ the first condition is not satisfied since $\widetilde{\alpha}_0 >$ 1.■

Opting Out Equilibria

An *Opting Out Profile* is characterized by strong types taking their outside opportunities at some time with probability 1, leaving weak types to play as in the complete information War of Attrition from that time on.

Define as $\overline{T}$ the natural number that solves:

$$\frac{H}{H+G} + (\pi_0 - \frac{H}{H+G})(\frac{1}{1-H-G})^{t-1} \leq 1 - G \leq \frac{H}{H+G} + (\pi_0 - \frac{H}{H+G})(\frac{1}{1-H-G})^{t}.$$

The next proposition shows that, if the probability of facing a weak type is relatively high, the optimal strategy of a strong type is such that she opts out at period $\overline{T} \geqslant 0$ with probability 1, and the optimal strategy of a weak type, from time $\overline{T}$ on, is to concede with a constant probability.

Proposition 3. *If* $\pi_0 \in \left(\frac{H}{H+G}, 1-G\right)$, *the unique SPBE is such that:*

$$\alpha_t = \frac{H}{\frac{H}{H+G}+(\pi_0-\frac{H}{H+G})\left(\frac{1}{1-H-G}\right)^t}, \quad \forall t \leq \overline{T},$$
$$\beta_t = \frac{G}{1-\left[\frac{H}{H+G}+(\pi_0-\frac{H}{H+G})\left(\frac{1}{1-H-G}\right)^t\right]}, \quad \forall t < \overline{T},$$

$\beta_t = 1$ *and* $\alpha_{t+1} = \alpha = \frac{a(1-\delta)}{1-a-\delta a}$ $\forall t \geq \overline{T}$.

And if $\pi_0 \in [1-G, 1)$ $\beta_t = 1$ $\forall t \geqslant 0$ *and* $\alpha_0 = \frac{a(1-\delta\pi_0)}{(1-a-\delta a)\pi_0}$, $\alpha_t = \alpha = \frac{a(1-\delta)}{(1-a-\delta a)}$ $\forall t \geq 1$.

Proof. See Appendix.■

In an Opting Out Equilibrium, weak types place a small probability of concession at each period. The posterior of facing a weak type opponent π_t increases over time, but the probability α_t that weak types concede decreases. In equilibrium, there will be some time $t = \overline{T}$ such that the optimal concession probability of the weak types cannot induce strong types to stay in the game beyond $\overline{T}$ since the payoff they get by opting out at that time, b, is greater than the expected payoff of waiting an aditional period for $(1 - a)$. After $\overline{T}$ the posterior probability of facing a weak opponent is 1. Players that are still at the negotiation table recognize themselves as weak types and thus, from that period $\overline{T}$ on, they play the Symmetric Perfect Equilibrium of the complete information War of Attrition without outside options. In this continuation the equilibrium concession probability remains constant over time at $\alpha_{t+1} = \frac{a(1-\delta)}{1-a-\delta a}$ $\forall t \geq \overline{T}$.

On the other hand, Proposition 3 also tells us that if the initial probability of facing a weak type is close to 1, that is, if $\pi_0 \in [1 - G, 1)$, then, in equilibrium, strong types opt out with probability 1 at $\overline{T} = 0$. In this case, even if the probability of facing a weak opponent is very high, the probability of receiving the preferred agreement is sufficiently low to make it worthwhile for a strong type to leave the negotiation immediately.

In an Opting Out Equilibrium players try to screen each other´s type by prolonging the game and thus imposing a delay cost on the opponent, as well as on themselves. After some time, strong types are convinced that they will never receive their preferred agreement and decide to opt out. From that moment on, nothing can convince players that the other will ever concede for sure, and thus they adopt the symmetric equilibrium strategies of the classical War of Attrition.

Pooling Equilibrium

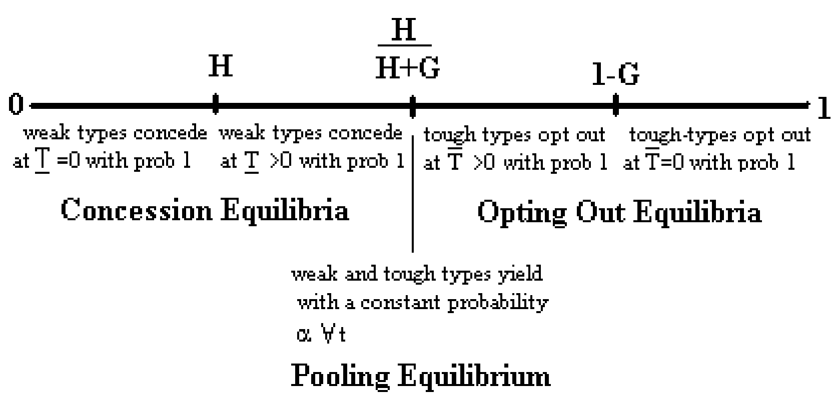
The next proposition establishes the unique combination of parameters (a, b, δ, π_0) for which the SPBE is pooling. Players follow strategies such that both types randomize at the same constant rate between yielding and not yielding. Therefore, there is no learning and $\underline{T} = \infty$.

Proposition 4. *If* $\pi_0 = \frac{H}{H+G}$, *the unique symmetric PBE is* $\alpha_t = \beta_t = H + G$, $\forall t \geq 0$.

Proof. See Appendix.■

If the probability of facing a weak opponent is exactly $\pi_0 = \frac{H}{H+G}$, in equilibrium, both types remain indifferent about conceding and opting out at every time. That is, in terms of randomized strategies, each player believes, at each time, that the probabilities that the opponent concedes or opts out at subsequent times are exactly so as to make continuation marginally worthwhile. No information is revealed along this equilibrium. No player updates his beliefs about the weakness of her opponent since if players concede and opt out at each time with the same probability, the posterior π_t is constant over time.

The next table summarizes our results so far:



Our characterization of the unique SPBE allows meaningful comparative statics results. We carry out this exercise for the limit, as $\delta \to 1$. This is the object of the next section.

3 Comparative Statics.

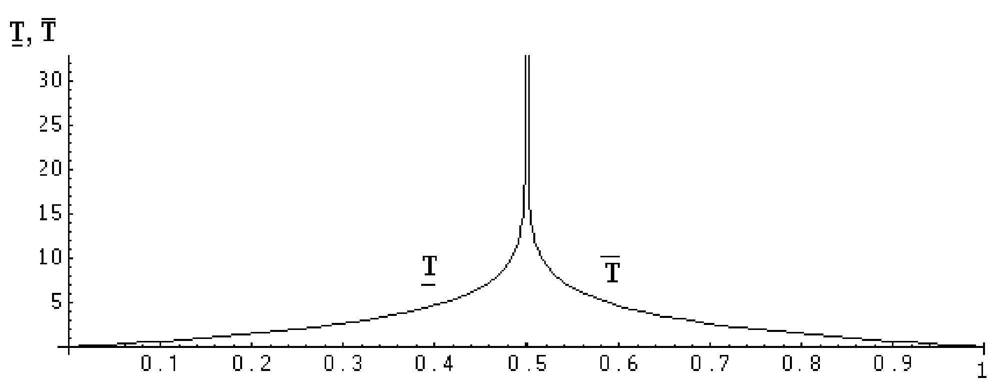
In this section we conduct comparative statics by analyzing the effects of change in the parameters in the limit of the game as the interval between periods becomes arbitrarily small. Let the length of each period in real time be denoted by Δ, $0 << 1$(there are $\frac{1}{\Delta}$ periods per unit of time), so that we can replace the term δ by $e^{-\Delta}$. We are interested in the limit of SPBE as $\Delta \to 0$.

It is easily checked that $\frac{H}{H+G}$ is independent of Δ and that $\lim\limits_{\Delta\to 0} H = 0$, $\lim\limits_{\Delta\to 0} G = 0$. The limit period $\underline{T}$, beyond which weak types will never continue in the negotiation in a Concession Equilibrium (see Proposition 2) and the limit period $\overline{T}$, beyond which strong types will surely opt out in an Opting Out Equilibrium (see Proposition 3), are given as functions[4] of the parameters of the game (a, b, π_0) as:

$$\underline{T}(\pi_0, a, b) = \frac{-a(1-a-b)}{(b-a(1-a))} \ln\left[\frac{b(a-\pi_0)+a\pi_0(1-a)}{ab}\right], \text{ for } \pi_0 \in \left(0, \frac{H}{H+G}\right),$$

$$\overline{T}(\pi_0, a, b) = \frac{-a(1-a-b)}{(b-a(1-a))} \ln\left[\frac{b(\pi_0-a)-a\pi_0(1-a)}{(1-a)(b-a)}\right] \text{ for } \pi_0 \in \left(\frac{H}{H+G}, 1\right).$$

Next figure displays $\underline{T}$ and $\overline{T}$ as functions of π_0 for a representative case ($a = \frac{1}{4}$, $b = \frac{3}{8}$).



[4]See appendix for the derivation of these functions.

We want to evaluate how $\underline{T}$ and $\overline{T}$ change as the result of changes in the parameters (a, b, π_0). We carry out this exercise in order to measure the effect on those parameters changes in the efficiency. We conjecture that efficiency improves as $\underline{T}$ decreases and $\overline{T}$ increases. A general proof for this conjecture is work in progress.

Next proposition establishes that an increase in the likelihood that the opponent has a valuable outside option reduces $\underline{T}$ and increases $\underline{T}$.

Proposition 5. *$\underline{T}$ decreases and $\overline{T}$ increases as π_0 decreases.*

Proof. See Appendix.■

Next we will analyze the effect of an increase of a on the limit periods $\underline{T}$ and $\underline{T}$. Define the following sets of parameters:

$$S_1 = \left\{(a,b) \text{ such that } b \leq (1-a)^2\right\},$$

$$S_2 = \left\{(a,b) \text{ such that } b > (1-a)^2 \text{ and } \frac{a(1-a-b)(b-a^2)}{(1-a)(b-a)(b-(1-a)^2)} > 1\right\},$$

$$S_3 = \left\{(a,b) \text{ such that } b > (1-a)^2 \text{ and } 0 < \frac{a(1-a-b)(b-a^2)}{(1-a)(b-a)(b-(1-a)^2)} < 1\right\}.$$

Let $x = \frac{b(\pi_0 - a) - a\pi_0(1-a)}{(1-a)(b-a)}$ and $\widetilde{x}$ be the solution to:

$$(b-(1-a)^2)\ln[x] + \tfrac{a(1-a-b)(b-a^2)}{(1-a)(b-a)}(\frac{1}{x} - 1) = 0. \qquad (2)$$

Proposition 6. *(i) $\underline{T}$ decreases as a increases. (ii) $\overline{T}$ increases as a increases $\forall (a,b) \in S_1 \cup S_2$. If $(a,b) \in S_3$, then $\frac{\partial \overline{T}}{\partial a} \geq 0$ if $x \in \left(0, \widetilde{x}\right]$ and $\frac{\partial \overline{T}}{\partial a} < 0$ if $x \in (\widetilde{x}, 1)$.*

Proof. See Appendix.■

We see that efficiency improves as a increases if a and b are close since an increase on a reduces the time at which weak types concede with probability 1 in a Concession Equilibrium, and increases the time at which strong types opt out with probability 1 in an Opting Out Equilibrium. However, the effect of the concession payoff a on $\overline{T}$ when a and b are far, depends on the relationship between a, b and π_0. In the next example we find the initial probability π_0^* such that $\frac{\partial \overline{T}}{\partial a} \geqslant 0$ if $\pi_0 \in \left(\frac{H}{H+G}, \pi_0^*\right]$ and $\frac{\partial \overline{T}}{\partial a} < 0$ if $\pi_0 \in (\pi_0^*, 1)$.

a	b	π_0^*
$\frac{1}{3}$	$\frac{58}{100}$	0.722556
$\frac{1}{3}$	$\frac{6}{10}$	0.631353
$\frac{1}{3}$	$\frac{62}{100}$	0.562209

If the value of the outside option is only slightly greater than the concession payoff, then the range of probabilities π_0 for which an increase of the size of the concession payoff improves efficiency is bigger.

Finally we analyze the effect of an increase in the value of the outside option, b, on $\underline{T}$ and $\overline{T}$. Define $y = \frac{b(a-\pi_0)+a\pi_0(1-a)}{ab}$ and let $\widetilde{y}$ the solution to:

$$\ln[y] + \tfrac{a(1-a-b)}{b(1-a)}(\frac{1}{y} - 1) = 0. \qquad (2)$$

Proposition 7. *(i) $\overline{T}$ increases as b increases. (ii) $\frac{\partial \underline{T}}{\partial b} \leq 0$ if $y \in \left(0, \widetilde{y}\right]$ and $\frac{\partial \underline{T}}{\partial b} > 0$ if $y \in (\widetilde{y}, 1)$.*

Proof. See Appendix.■

If the value of the outside option increases, strong types take longer to opt out with probability 1 in an Opting Out Equilibrium. However, the effect of b on $\underline{T}$, is not clear cut. In this case the sign of this derivative will depend on the relationship between a, b and π_0. Since is not possible to find an analytical solution to the equation (2), we make some numerical computations. Notice that finding $\widetilde{y}$ is equivalent to find the initial probability π_0^* such that $\frac{\partial \underline{T}}{\partial b} < 0$ if $\pi_0 \in (0, \pi_0^*]$ and $\frac{\partial \underline{T}}{\partial b} > 0$ if $\pi_0 \in (\pi_0^*, \frac{H}{H+G})$. The following table shows some numerical examples:

a	b	π_0^*
$\frac{1}{3}$	$\frac{2}{5}$	0.638251
$\frac{1}{4}$	$\frac{2}{5}$	0.413369
$\frac{1}{5}$	$\frac{2}{5}$	0.301117

We see that the difference between a and b matters. If the value of the outside option is only slightly greater than the concession payoff, then the range of probabilities π_0 for which an increase of the size of the outside option improves efficiency is bigger.

4 Conclusions

In this paper we have explored the effect of the private information about outside options on the outcomes of negotiations. In order to address this issue we analyzed a War of Attrition allowing players to leave the negotiation in order to opt out and we characterized the Symmetric Perfect Bayesian Equilibrium of this game. There are two types of players: a weak type who has a valueless outside option-she always prefers conceding rather than opting out- and a strong type who has a valuable outside option that she prefers to take rather than conceding. We show that uncertainty about the possibility that the opponent opts out improves efficiency, since it increases the equilibrium probability of concession. More precisely, if the probability that the opponent is strong is relatively high, in equilibrium, the negotiation eventually ends with a sure concession. In these cases, we are able to identify a time $\underline{T}$ at which a player with a valueless outside option, will concede with probability 1, and a player with an outside option will wait to obtain a concession until $\underline{T}+1$; then, she will opt out with probability 1. On the other extreme, if the likelihood of a weak opponent is high, strong types stay in the game for a while and eventually leave the negotiation and opt out with probability 1. From that date $\overline{T}$ on, weak types play the (inefficient) symmetric equilibrium of the classical War of Attrition with complete information. Even in this case, the probability of concession by weak types along the uncertainty phase of the equilibrium play increases.

Proof of Proposition 1

A pure strategy for player i type $\tau = W, S$, is a time t_i^τ at which she plans to yield (to concede is she is weak and to opt out is she is strong) given than no player yields before that time. If a pure SPBE exists, then $t_i^W = t_j^W = t_W$ and $t_i^S = t_j^S = t_S$. Assume that $t_S \leq t_W$. Thus, strong types know that, in equilibrium, weak types do not concede before they opt out with certainty. Then, it is optimal for a strong type to opt out at period 0, so she avoids any discounting of the payoff. The same happens to a weak type, since she knows she is not going to get any concession from her opponent. Thus, if there is a SPBE with $t_S \leq t_W$, it must be $t_W = t_S = 0$. But this cannot be an equilibrium since strong types will deviate from this strategy by delaying at least one period the decision of opting out since $b < (1-a)\pi_0 + b(1-\pi_0)$.

The other potential equilibrium is $t_W < t_S$ in which case $t_W = 0$ and $t_S = x$ with $x \geq 1$. If weak types concede in equilibrium at t=0, then it must be true that $a \geq (1-a)\pi_0 + a\delta(1-\pi_0)$ or $\pi_0 \leq \frac{a(1-\delta)}{1-a-\delta a}$. Since $\pi_0 \leq \frac{a(1-\delta)}{1-a-\delta a} < \frac{b(1-\delta)}{1-a-\delta b}$, strong types deviate and opt out at t=0 since $b > (1-a)\pi_0 + b\delta(1-\pi_0)$.■

Proof of Proposition 2b

Consider the equation that rules the posterior:

$$\pi_t = \frac{H}{H+G} + (\pi_0 - \frac{H}{H+G})\left(\frac{1}{1-H-G}\right)^t.$$

If $\pi_0 < \frac{H}{H+G}$, π_t is decreasing over time and, thus $\alpha_t = \dfrac{H}{\pi_t}$ increases. At some period t, α_t reaches the value of 1. We denote that time as $\underline{T}$. In order to identify the time $\underline{T}$ we must use:

$$\pi_{\underline{T}-1}\alpha_{\underline{T}-1} = H,$$

$$\pi_{\underline{T}}\alpha_{\underline{T}} \leq H.$$

Since $\alpha_{\underline{T}} = 1$ and $\alpha_{\underline{T}-1} < 1$ then $\pi_{\underline{T}-1} \geq H \geq \pi_{\underline{T}}$. Using the solution for π_t, $\underline{T}$ will be the natural number that solves:

$$\tfrac{H}{H+G} + (\pi_0 - \tfrac{H}{H+G})(\tfrac{1}{1-H-G})^t \leq H \leq \tfrac{H}{H+G} + (\pi_0 - \tfrac{H}{H+G})(\tfrac{1}{1-H-G})^{t-1}.$$

By lemma 1 we know that if $\alpha_{\underline{T}} = 1$ then $\beta_{\underline{T}+1} = 1$.$\blacksquare$

Proof of Proposition 3

If $\frac{H}{H+G} < \pi_0 < 1-G$, then π_t is increasing over time and β_t increases until, at some point, it reaches the value of 1. We denote as $\overline{T}$ that time and

$$(1 - \pi_{T_H^*-1})\beta_{\overline{T}-1} = G,$$

$$(1 - \pi_{\overline{T}})\beta_{\overline{T}} \leq G.$$

Since $\beta_{\overline{T}} = 1$ and $\beta_{\overline{T}-1} < 1$, then $1 - \pi_{\overline{T}-1} \geq G \geq 1 - \pi_{\overline{T}}$. Using the solution of π_t, $\overline{T}$ will be the natural number that solves:

$$\tfrac{H}{H+G} + (\pi_0 - \tfrac{H}{H+G})(\tfrac{1}{1-H-G})^{t-1} \leq 1 - G \leq \tfrac{H}{H+G} + (\pi_0 - \tfrac{H}{H+G})(\tfrac{1}{1-H-G})^t.$$

Since $\beta_{\overline{T}} = 1$, then $\pi_t = 1\ \forall t \geq \overline{T} + 1$. Players that are still playing are weak types and thus $\alpha_t = \frac{a(1-\delta)}{1-a-\delta a}$ for $t \geqslant \overline{T} + 1$.

If $1 - G \leqslant \pi_0$, the SPBE is $\beta_t = 1\ \forall t \geq 0$ and $\alpha_0 = \dfrac{a(1-\delta\pi_0)}{(1-a-\delta a)\pi_0}$ $\alpha_t = \dfrac{a(1-\delta)}{1-a-\delta a}$ $\forall t \geq 1$. Then, a weak type will optimally randomize between conceding and not conceding at each t if:

$$a = \pi_0\alpha_0(1-a) + a\delta(1 - \pi_0\alpha_0 - (1-\pi_0)\beta_0),$$

$$a = \pi_t\alpha_t(1-a) + a\delta(1 - \pi_t\alpha_t - (1-\pi_t)\beta_t)\ \forall t \geq 1.$$

Given these strategies, $\pi_t = 1\ \forall t \geq 1$. We substitute $\beta_t = 1\ \forall t \geq 0$ and $\alpha_0 = \dfrac{a(1-\delta\pi_0)}{(1-a-\delta a)\pi_0}$ $\alpha_t = \dfrac{a(1-\delta)}{1-a-\delta a}$ in those equations and check if they are satisfied $\forall t$.

Consider now a strong type. Given the opponent's strategy, $\beta_t = 1$ $\forall t \geq 0$ and $\alpha_0 = \dfrac{a(1-\delta\pi_0)}{(1-a-\delta a)\pi_0}$ $\alpha_t = \dfrac{a(1-\delta)}{1-a-\delta a}$ $\forall t \geq 1$ he will opt out with probability 1 from period 0 on if:

$$b > \pi_0\alpha_0(1-a) + b(1-\pi_0)\beta_0 + b\delta(1-\pi_0\alpha_0 - (1-\pi_0)\beta_0) = 0,$$

$$b > \pi_t\alpha_t(1-a) + b(1-\pi_t)\beta_t + b\delta(1-\pi_t\alpha_t - (1-\pi_t)\beta_t) \text{ for } t \geq 1$$

Since $\pi_t = 1$ $\forall t \geq 1$, the second condition is satisfied if $b > \alpha_t(1-a) + b\delta(1-\alpha_t)$. Substituting α_t,

$$b > \frac{a(1-\delta)}{1-a-\delta a}(1-a) + b\delta(1 - \frac{a(1-\delta)}{1-a-\delta a}).$$

Or $\frac{b(1-\delta)}{1-a-\delta b} > \frac{a(1-\delta)}{1-a-\delta a}$ that is true since $b > a$.

At t=0 it must be satisfied that $b > \pi_0(1-a)\alpha_0 + b(1-\pi_0) + b\delta\pi_0(1-\pi_0\alpha_0)$. Substituting α_0 and β_0 it is easy to check that this equation is satisfied only if $\pi_0 \geq 1-G$.

Now we will prove that if $\pi_0 \geq 1-G$, the unique symmetric SPBE is $\{\{\alpha_t\}_0^\infty \{\beta_t\}_0^\infty\}$ such that $\beta_t = 1$ $\forall t \geq 0$ and $\alpha_0 = \dfrac{a(1-\delta\pi_0)}{(1-a-\delta a)\pi_0}$, $\alpha_t = \dfrac{a(1-\delta)}{1-a-\delta a}$ $\forall t \geq 1$. We will explore all possible candidates and see that, indeed, this is the unique SPBE.

First, consider a SPBE $\left\{\left\{\hat{\alpha}_t\right\}_0^\infty, \left\{\hat{\beta}_t\right\}_0^\infty\right\}$ such that $0 < \hat{\alpha}_t < 1$ and $0 < \hat{\beta}_t < 1$ $\forall t \geq 0$. Then,

$$a = (1-a)\pi_t\,\hat{\alpha}_t + a\delta(1-\pi_t\,\hat{\alpha}_t - (1-\pi_t)\,\hat{\beta}_t),$$

$$b = (1-a)\pi_t\,\hat{\alpha}_t + b(1-\pi_t)\,\hat{\beta}_t + b\delta(1-\pi_t\,\hat{\alpha}_t - (1-\pi_t)\,\hat{\beta}_t),$$

for $\forall t \geq 0$. At t=0 these conditions are rewritten as:

$$\hat{\alpha}_0\,\pi_0 = H,$$

$$\hat{\beta}_0 (1-\pi_0) = G.$$

But since $\pi_0 \geq 1-G$, $\hat{\beta}_0 \geq 1$ contradicting the assumption that $0 < \hat{\beta}_t < 1$ $\forall t \geq 0$.

Second, assume that there is a SPBE such that $\hat{\beta}_t = 1$ $\forall t \geq 0$ and $0 < \hat{\alpha}_t < 1$ such that $\hat{\alpha}_0 \neq \dfrac{a(1-\delta\pi)}{(1-a-\delta a)\pi}$, $\hat{\alpha}_t \neq \dfrac{a(1-\delta)}{1-a-\delta a}$ $\forall t \geq 1$. Notice that if $\hat{\beta}_0 = 1$ and $0 < \hat{\alpha}_0 < 1$ then $\pi_1 = 1$. But this cannot be an equilibrium since a weak type will deviate and concede with probability 1 at $t=1$ if $\hat{\alpha}_1 < \frac{a(1-\delta)}{1-a-\delta a}$ since $a > (1-a)\hat{\alpha}_1 + a\delta(1-\hat{\alpha}_1)$ and will never concede if $\hat{\alpha}_t > \frac{a(1-\delta)}{1-a-\delta a}$. The same happens at t=0.

And finally, assume that there is a SPBE with $0 < \hat{\beta}_t < 1$ $\forall t \geq 0$ and $\hat{\alpha}_0 = \dfrac{a(1-\delta\pi_0)}{(1-a-\delta a)\pi_0}$, $\hat{\alpha}_t = \dfrac{a(1-\delta)}{1-a-\delta a}$ $\forall t \geq 1$. In that case, at t=0, it must be true that:

$$a = (1-a)\pi_0 \hat{\alpha}_0 + a\delta(1-\pi_0 \hat{\alpha}_0 - (1-\pi_0)\hat{\beta}_0)$$

$$b = (1-a)\pi_0 \hat{\alpha}_0 + b(1-\pi_0)\hat{\beta}_0 + b\delta(1-\pi_0 \hat{\alpha}_0 - (1-\pi_0)\hat{\beta}_0)$$

Substituting $\hat{\alpha}_0 = \dfrac{a(1-\delta\pi_0)}{(1-a-\delta a)\pi_0}$ in the first condition, it must be that $\hat{\beta}_0 = 1$ contradicting that $0 < \hat{\beta}_t < 1$ $\forall t \geq 0$.$\blacksquare$

Proof of Proposition 4

Now consider the case $\pi_0 = \frac{H}{H+G}$. Then $\pi_t = \pi_0$ and $\alpha_t = \dfrac{H}{\pi_0} = H+G$ and $\beta_t = \frac{G}{1-\pi_0} = H+G$ $\forall t \geq 0$.$\blacksquare$

The derivation of $\underline{T}$ and $\overline{T}$.

We reduce the length of each period to $0 < \Delta < 1$ (there are $\frac{1}{\Delta}$ periods per unit of time) and the term δ is replaced by $e^{-\Delta}$. Define:

$$H^{'} = \frac{ab(1-e^{-\Delta})}{ae^{-\Delta}(1-a-e^{-\Delta}b)+b(1-e^{-\Delta})(1-a-e^{\Delta}a)}$$

$$G^{'} = \frac{(1-e^{-\Delta})(1-a)(b-a)}{ae^{-\Delta}(1-a-e^{-\Delta}b)+b(1-e^{-\Delta})(1-a-e^{-\Delta}a)}$$

$$\frac{H'}{H'+G'} = \frac{ab}{b-a(1-a)}$$

$\frac{H^{'}}{H^{'}+G^{'}}$ is independent of Δ. It is easily checked that $\lim_{\Delta\to 0} H^{'} = 0$ and $\lim_{\Delta\to 0} 1-G = 1$..

Proposition 2 establishes that we can identify an ending period $\underline{T}$ at which the equilibrium probability of conceding is 1 if $\pi_0 \in \left(H^{'}, \frac{H^{'}}{H^{'}+G^{'}}\right)$. This $\underline{T}$ is the natural number that solves:

$$\tfrac{H}{H+G} + (\pi_0 - \tfrac{H}{H+G})(\tfrac{1}{1-H-G})^{\frac{t}{\Delta}} \leq H \leq \tfrac{H}{H+G} + (\pi_0 - \tfrac{H}{H+G})(\tfrac{1}{1-H-G})^{\frac{t}{\Delta}-\Delta}.$$

Or, for each possible expected delay $\underline{T}$ we have a compatible interval of π

$$\pi_0 \in \left(\tfrac{H^{'}}{H^{'}+G^{'}}(1-(1-H^{'}-G^{'})^{\frac{t}{\Delta}}), \tfrac{H^{'}}{H^{'}+G^{'}}(1-(1-H^{'}-G^{'})^{\frac{t}{\Delta}+\Delta}\right].$$

The size of this interval tends to 0 as $\Delta \to 0$. Hence, in the limit, we have a function

$$\pi_0 = \frac{H^{'}}{H^{'}+G^{'}}\left[1-e^{-tI}\right]$$

with $I = \frac{b-a(1-a)}{a(1-a-b)}$. Or, given the parameters of the game (a, b, π_0)

$$\underline{T} = \frac{-1}{I}\ln(1-\tfrac{H^{'}+G^{'}}{H^{'}}\pi_0)$$

We consider now the interval of probabilities $\pi_0 \in \left(\frac{H^{'}}{H^{'}+G^{'}}, 1-G^{'}\right)$. Proposition 4 shows that, in equilibrium, strong types won't remain in the game beyond some period $\overline{T}$ that can be identified as the natural number that solves:

$$\frac{H}{H+G}+(\pi_0-\frac{H}{H+G})(\frac{1}{1-H-G})^{\frac{t}{\Delta}-\Delta}\leq 1-G\leq \frac{H}{H+G}+(\pi_0-\frac{H}{H+G})(\frac{1}{1-H-G})^{\frac{t}{\Delta}}$$

We compute the interval of probabilities for which $\overline{T}=\frac{t}{\Delta}$,

$$\pi_0\in\left[\frac{H'}{H'+G'}+\frac{G'}{H'+G'}(1-H'-G')^{\frac{t}{\Delta}+\Delta},\frac{H'}{H'+G'}+\frac{G'}{H'+G'}(1-H'-G')^{\frac{t}{\Delta}}\right]$$

As Δ goes to 0 the size of this interval tends also to 0 and

$$\pi_0=\frac{H'}{H'+G'}+\frac{G'}{H'+G'}e^{-tI}$$

Therefore

$$\overline{T}=\frac{-1}{I}\ln\left[\pi_0(\frac{H'+G'}{G'})-\frac{H'}{G'}\right]$$

Proof of Proposition 5

We simply compute the partial derivatives of $\underline{T}$ and $\overline{T}$ with respect to π_0. Denote as $y=\frac{b(a-\pi_0)+a\pi_0(1-a)}{ab}$ and $x=\frac{b(\pi_0-a)-a\pi_0(1-a)}{(1-a)(b-a)}$. Then,

$$\frac{\partial\underline{T}}{\partial\pi_0}=\frac{(1-a-b)}{by}>0,$$

$$\frac{\partial\overline{T}}{\partial\pi_0}=\frac{-a(1-a-b)}{(1-a)(b-a)x}<0,$$

since $0<y<1$, $0<x<1$ and $a<b<1-a$.■

Proof of Proposition 6

The partial derivative $\frac{\partial\underline{T}}{\partial a}$ is,

$$\frac{\partial\underline{T}}{\partial a}=\frac{1}{(b-a(1-a))^2}\left[b(b-(1-a)^2)\ln[y]-(1-a-b)(b-a^2)(\frac{1}{y}-1)\right].$$

In order to prove that $\frac{\partial T}{\partial a} < 0$ we will consider two cases:

(i) $b \geqslant (1-a)^2$. The sign of the derivative is clearly negative since $0 < y < 1$ and $a < b < 1-a$.

(ii) $b < (1-a)^2$. We study the function

$$F(y) = b(b-(1-a)^2)\ln[y] - (1-a-b)(b-a^2)(\frac{1}{y}-1)$$

It is easy to check that $F(1) = 0$, $F(0) = -\infty$ and $F(y)$ has a maximum on $y^* = \frac{-(1-a-b)(b-a^2)}{b^2-b(1-a)^2}$. Since $\frac{-(1-a-b)(b-a^2)}{b^2-b(1-a)^2} > 1$ then $F(y) < 0$ $\forall y \in (0,1)$.

The derivative $\frac{\partial \overline{T}}{\partial a}$ is

$$\frac{\partial \overline{T}}{\partial a} = \frac{b}{(b-a(1-a))^2}\left[(b-(1-a)^2)\ln[x] + \frac{a(1-a-b)(b-a^2)}{(1-a)(b-a)}(\frac{1}{x}-1)\right]$$

We study the sign of this derivative and find two cases:

(i) $b \leq (1-a)^2$. Clearly $\frac{\partial \overline{T}}{\partial a} > 0$ since $0 < x < 1$ and $a < b < 1-a$.

(ii) $b > (1-a)^2$. The sign of $\frac{\partial \overline{T}}{\partial a} = signF(x)$ with $F(x) = (b-(1-a)^2)\ln[x] + \frac{a(1-a-b)(b-a^2)}{(1-a)(b-a)}\left[\frac{1}{x}-1\right]$. This function has a minimum at $x^* = \frac{a(1-a-b)(b-a^2)}{(1-a)(b-a)(b-(1-a)^2)}$ since $F''(x^*) > 0$ and takes the values $F(0) = +\infty$ and $F(1) = 0$. Thus, if $x^* > 1$, then $\forall x \in (0,1)$ $F(x) > 0$. Otherwise, if $x^* < 1$, then $\frac{\partial \overline{T}}{\partial a} > 0$ $\forall x \in (0, \widetilde{x})$ and $\frac{\partial \overline{T}}{\partial a} < 0$ $\forall x \in (\widetilde{x}, 1)$ where $\widetilde{x}$ is the unique root of $F(x)$ on the range $x \in (0,1)$.■

Proof of Proposition 7

First, we compute the partial derivative of $\overline{T}$ with respect to b:

$$\frac{\partial \overline{T}}{\partial b} = \frac{a(1-a)}{r(b-a(1-a))^2}\left[(1-a)\ln[x] - \frac{a^2(1-a-b)}{(b-a)(1-a)}(\frac{1}{x}-1)\right] < 0,$$

Now we derive $\frac{\partial \underline{T}}{\partial b}$ that is,

$$\frac{\partial \underline{T}}{\partial b} = \frac{a(1-a)^2}{r(b-a(1-a))^2}\left[\ln[y] + \frac{a(1-a-b)}{b(1-a)}(\frac{1}{y}-1)\right].$$

It is clear that $sign\frac{\partial \underline{T}}{\partial b} = signJ(y)$ with $J(y) = \ln[y] + \frac{a(1-a-b)}{b(1-a)}(\frac{1}{y}-1)$.

This function takes values $J(0) = +\infty$ and $J(1) = 0$ and its derivative $J^{'}(y^*) = 0$ with $y^* = \frac{a(1-a-b)}{b(1-a)}$. It is easy to check that $J(y^*) < 0$ and that $J(y)$ is decreasing on $(0, y^*)$ and increasing on $(y^*, 1)$. Since $\frac{a(1-a-b)}{b(1-a)} < 1$, then $J(y)$ has a unique root on the range $y \in (0, 1)$. This root is the $\widetilde{y}$ that solves the equation

$$\ln[y] + \tfrac{a(1-a-b)}{b(1-a)}(\frac{1}{y} - 1) = 0.$$

Then, $J(y) > 0$ if $y \in (0, \widetilde{y})$ and $J(y) < 0$ if $y \in (\widetilde{y}, 1)$.■

Bibliography

Binmore K., A. Shaked. and J. Sutton (1989). "An Outside Option Experiment". *The Quarterly Journal of Economics*, Nov 1989, 753-770.

Chatterjee, K. and L. Samuelson (1987). "Bargaining with Two-sided Incomplete Information: An Infinite Horizon Model with Alternating Offers". *Review of Economic Studies,* 54, 175-192.

Compte O. and P. Jehiel (2000). "On the Role of Outside Options in Bargaining with Obstinate Parties". forthcoming in *Econometrica*.

Fudenberg, D., D. K. Levine and J. Tirole (1987). "Incomplete Information Bargaining with Outside Opportunities". *The Quarterly Journal of Economics*, 37-50.

Fudenberg, D and J. Tirole (1986). "A Theory of Exit in Duopoly". *Econometrica* Vol.54, n. 4, 943-960.

Hendricks, K., A. Weiss and C. Wilson (1988). "The War of Attrition in Continuous Time with Complete Information". *International Economic Review*, Vol. 29, n. 4, 663-680.

Osborne, M. J and A. Rubinstein. "Bargaining and Markets", 1990, Academic Press.

Ponsatí C. and J. Sakovics (1995). "The war of attrition with incomplete information". *Mathematical Social Sciences*, 29, 239-254.