The Value of Waiting to Learn

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Abstract

The goal of this paper is to study irreversible investment under incomplete information. We extend McDonald and Siegel's (1986) model to the case where the expected rate of return of the project cannot be observed but is known to be either low or high. Waiting and observing the realizations of the value of the project provides information to the investor who can update her beliefs about the true value of the expected return. Uncertainty increases the option value of waiting but damages the quality of the signal received. We show that beliefs follow a martingale and the optimal investment trigger depends on the degree of optimism. We obtain that the investment trigger frontier of a Bayesian investor lies below the one of a non-updating belief investor.

JEL classification: D81, D83, D92.

Keywords: Option Value, Learning, Uncertainty, Irreversible Investment.

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ABSTRACT

We introduce incomplete information into McDonald-Siegel's (1986) model of irreversible investment. The project average growth rate cannot be observed but is known to be constant, either low or high. Waiting and observing the realizations of the project value allows the investor to update her beliefs about the true average growth rate value. Uncertainty has an ambiguous effect as it increases the option value of waiting but damages the quality of the signal received. As far as investment timing is concerned, the investment trigger frontier of a Bayesian investor lies below the one of a non-updating belief investor.

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In the last two decades, the nature of the investment-uncertainty relationship has been at the center stage of the investment literature, building on some earlier works on investment by Jorgenson (1963) and Arrow (1968). In particular, scholars have focused their attention on adjustment costs and irreversibility. Within a continuous time framework, Abel (1983) shows that under perfect competition, in presence of symmetric convex adjustment costs, more uncertainty (as measured by a higher instantaneous variance of the output price) leads to an increase in the optimal rate of investment, provided that the profit function is convex in prices. Recently, Sarkar (2002) obtains that in a real option framework, an increase in uncertainty can raise the probability of investing. Caballero (1991) identifies the nature of competition as the key determinant of the relationship and shows that under imperfect competition, the investment-uncertainty relationship can become negative when the adjustment costs are highly asymmetric and the marginal profitability of capital is sharply decreasing in the level of capital. Allowing for partial reversibility, Abel and Eberly (1994) extend the Jorgensonian concept of user cost to the case of uncertainty when a firm can purchase and sell some capital at different prices. Abel and Eberly (1996) develop a "unified model of investment under uncertainty" incorporating both adjustment costs and irreversibility. They obtain three different regimes: Positive investment, no investment and disinvestment. The value of Tobin's q, the shadow price of installed capital determines the regime.

McDonald and Siegel (1986) were among the first to study the implications of irreversibility on the timing of investment decisions. Since then, an extensive literature in real options has emphasized the benefits from delaying the timing of undertaking an irreversible investment. The seminal book, Investment under Uncertainty, by Dixit and Pindyck (1994) represents a comprehensive review on real options. The usual way of introducing uncertainty is to assume that the value of a project (or some other economic indicator) follows a given (stochastic) law of motion *known* by investor. Irreversibility implies that the optimal decision is to wait until the value of the project hits a threshold (investment trigger) which can be significantly higher that the cost of investing. The reason is that the decision maker recognizes that investing means killing her option (and giving up the benefits of delaying investment), so the associated opportunity cost should be taken into account.

However, in many real life investment opportunities, the characteristics of a project are hardly known with perfect accuracy. Applications include development projects in new and unfamiliar markets, research and development, investing into start-up companies. The objective of this paper is to incorporate incomplete information into the traditional real option framework and shed some additional light on the role of uncertainty in irreversible investment decisions when learning takes place.

This paper is at the crossroad of two literatures: Real options and the value of information. McDonald and Siegel (1986) highlight "the value of waiting to invest". In this paper, we aim at characterizing *the value of waiting to learn (and invest)*. Some of the central issues of this paper are related to the work by Bernanke (1983) and Venezia (1983). Bernanke (1983) points out that only unfavorable outcomes actually matter for undertaking or postponing an investment. In other words, the distribution of payoffs can be truncated and actually, only the left tale is to be considered. He calls this effect the "bad news principle of irreversible investment". Venezia (1983) examines the case of a firm that can sell an asset whose value is observable but its mean value is unknown. He obtains that a Bayesian manager has more incentive to keep the asset than a manager who does not revise her beliefs. Thus, the former has a higher reservation price than the latter. Bernardo and Chowdhry (2002) consider the case of a firm that can learn about its own resources and decide in an irreversible fashion to either to exit the market (i), scale up its existing business (ii) or diversify its activities (iii). They show that the firm chooses options (i) or (ii) if it accesses its resources to be low enough. Conversely the firm chooses option (iii) when resources are thought to be high enough. In the between, the firm keeps on experimenting. Demers (1991) considers a risk neutral firm that is uncertain about the state of demand and updates its beliefs using Bayes' rule. He shows that irreversibility and anticipation of receiving information signal in the future lead to a more cautious investment behavior than under complete information. Cukierman (1980) investigates how a risk neutral firm selects projects among several investment opportunities. Increased uncertainty causes a decrease in the current level of investment by making it more profitable to wait longer for more information before choosing an investment project.

In this article, we characterize the value of the information and focus on the impact of optimism - defined as the beliefs that the investment project is "good", on an irreversible investment decision. Our information background is a continuous-time model of Bayesian learning a la Bolton and Harris (1999). In their paper, the authors derive the shadow value of experimenting. In this paper, the shadow value is interpreted as the value of time of waiting to learn. In a similar information framework, Keller and Rady (1999) and Cripps, Keller and Rady (2000) study optimal experimentation by a monopolist and a duopolist respectively. A closely related work by Moscarini and Smith (2001) consider the case of a decision maker who buys some information to improve the precision of a signal before undertaking some action. They show that the optimal experimentation level increases with a project's expected payoff.

The main contribution of the paper is to clarify the effects of learning and uncertainty on irreversible investment decisions in presence of incomplete information. We consider the case of a risk neutral investor who discounts future at a constant rate and knows that the expected return of the project is constant and can only take two values, high or low. By waiting, the investor can observe the realizations of the value of the project and update her beliefs about the project type nature. Alternatively, the framework of this paper is suitable for analyzing investment decision in natural resources as for instance investigated in Brennan and Schwartz (1985) when the convenience yield is not observable.

The role of uncertainty (as measured by the instantaneous volatility of the project) is now twofold. First, as in the classical model of irreversible investment, more uncertainty induces the investor to require a larger wedge between the value of the project that trigger investment and the cost of investing. Second, uncertainty also affects the speed at which the investor updates her beliefs. If there is a lot of uncertainty, observing the realizations of the value of the project provides little information and consequently the investor's beliefs evolve slowly. These two effects work in opposite directions on the option value. As in the usual model, due to the convexity of the payoff, uncertainty raises the option value of waiting but in addition it damages the quality of the signal, thus reducing the information value.

The model also allows us to study the impact of beliefs on investment decision under uncertainty. If the investor becomes more optimistic, i.e., assigning a high a probability that the average growth rate of the project is high, her option value increases which in turn delays her investment decision. The comparison between a Bayesian investor and a non-updating investor reveals that the former always invests at a lower threshold than the latter: Information accelerates investment measured in terms of threshold. As far as the timing of investment (measured by the average time until investment takes place) is concerned, two opposing effects compete. On the one hand, optimism increases the value of waiting, thus rising the investment trigger value. On the other hand, optimism increases the expected return of the investment (under the investor information structure) accelerating the investment decision. Numerical simulations show that the overall effect of optimism on the average time until investment depends on how far away the value of the project is from the optimal trigger investment frontier. The average time is decreasing as a function of optimism for small values of the project and then becomes hump shape as the value of the project rises and ultimately non-decreasing when the value of the project gets close to the investment threshold frontier.

The paper is organized as follows. Section 2 describes the economic setting and provides analytical results on the option value, the effect of optimism on the optimal investment frontier and the average time to invest. Section 3 displays some numerical simulations. Section 4 concludes. Proofs of all results are collected in the appendix.

1. The Economic Setting

We consider a standard irreversible investment problem. Time is continuous. A firm has to choose optimally the timing of its investment under uncertainty. The main innovation of the paper lies in the fact that the average growth rate (drift) of the project is unknown and waiting provides some information about the true value of the drift.

1.1. Investment opportunity and information structure

Uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, P^w)$ on which is defined a *one* dimensional (standard) Brownian motion w. A state of nature ω is an element of Ω. $\mathcal{F}$ denotes the tribe of subsets of Ω that are events over which the probability measure P^w is assigned.

A risk neutral investor has to choose when to invest into a project whose value V fluctuates across time according to a geometric Brownian motion

$$dV(t) = V(t)\left(\mu dt + \sigma dw(t)\right), \tag{1.1}$$

where $dw(t)$ is the increment of a standard Wiener process under P^w, μ represents the average growth rate of the project and σ captures the magnitude of the uncertainty.

The investment is irreversible with cost $I > 0$ and the future discount rate $r > 0$ is a constant. The parameter μ is *unknown* to the investor but the latter knows that μ can only take two values, h (high) or l (low). For technical reasons, we assume that $r > h > l > 0$.

An alternative formulation of the problem is the following. As in Brennan and Schwartz (1985), the project V can represent a mine whose reserves Q are constant and under a risk neutral probability measure P^w, the spot price S is given by

$$dS(t) = S(t)\left((r-\delta)dt + \sigma dw(t)\right).$$

r is the constant risk free rate and δ is the convenience yield rate that is not observable but known to be constant and can take only two possible values. Indeed, the value of the mine V satisfies the same dynamics as (1.1). Setting $\delta = r - \mu$ brings us back to the first formulation which is used in the sequel.

Even though an investor does not observe the true value for μ, she can observe the realizations across time of the project V and therefore infer the true value for the drift. Let $\mathcal{F}_t$ be the σ-algebra generated by the observations of the value of the project, $\{V(s); 0 \leq s \leq t)\}$ and augmented. At time t, the investor's information set is $\mathcal{F}_t$. The filtration $\mathbb{F} = \{\mathcal{F}_t,\ t \in \mathbb{R}_+\}$ is the information structure and satisfies the usual conditions (increasing, right-continuous, augmented). At time t, let $p(t)$ be the probability or the investor's beliefs that μ is equal to h, i.e., $p(t) = \Pr(\mu = h \mid \mathcal{F}_t)$. Using Bayes' rules, the evolution across time of the posterior probability p is given by the following lemma.

Lemma 1. *The law of motion of the posterior beliefs P is*

$$dp(s) = \frac{h-l}{\sigma} p(s)(1-p(s))d\overline{w}_p(s),$$

where

$$\begin{aligned} d\overline{w}_p(s) &= \frac{1}{\sigma V(s)}\left(dV(s) - E^P\left[dV(s) \mid \mathcal{F}_s\right]\right) \\ &= dw(s) + \frac{1}{\sigma}\left(\mu - (p(s)h + (1-p(s))l)\right)ds, \end{aligned}$$

is the increment of a standard Wiener process under P, relative to the filtration $\mathbb{F}$.

Proof.

See Liptser and Shiryaev, 2001 p 317 and for a more intuitive derivation see Bolton and Harris (1999). ■

Changes in beliefs are increasing in the wedge $h - l$: When the two average growth rates differ significantly more information can be obtained and the investor can revise her beliefs more quickly. Conversely, when the quality of the signal is poor (high value of σ) or when the investor is almost certain of the value of μ (p close to 0 or 1), little information can be extracted and therefore beliefs do not change much. In addition, p is a martingale under P relative to $\mathbb{F}$ so on average, the investor's beliefs does not change[1].

Let P_h and P_l be the probability measures under which the process V is a geometric Brownian motion with constant drift $\mu = h$ and $\mu = l$, respectively. For $\mu \in \{l, h\}$, define the processes $\gamma_{p,\mu}$ and $\xi_{p,\mu}$ by

$$\gamma_{p,\mu}(t) = \frac{\mu - (p(t)h + (1 - p(t))l)}{\sigma}$$

and

$$\xi_{p,\mu}(t) = \exp\left(-\int_0^t \gamma_{p,\mu}(s)dw(s) - \frac{1}{2}\int_0^t \gamma_{p,\mu}^2(s)ds\right).$$

Note that since $\gamma_{p,\mu}$ is a bounded process, $\xi_{p,\mu}$ is a martingale under $\mathbb{F}$ (Karatzas and Shreve, 1998, p. 17). Moreover, $\xi_{p,\mu}$ is the density process of the Radon-Nikodym derivative of P with respect to P_μ, i.e.,

$$\xi_{p,\mu}(t) = \frac{dP(t)}{dP_\mu(t)}.$$

It can be shown that when $\mu = h$, then $\xi_{p,h}(t) = \frac{p_0}{p(t)}$ and when $\mu = l$, then $\xi_{p,l}(t) = \frac{1-p_0}{1-p(t)}$.

Define $\phi_{p,l}(t) = \frac{p(t)}{1-p(t)}$ and $\phi_{p,h}(t) = \frac{1-p(t)}{p(t)}$. One can be easily check that under P_h and P_l respectively

$$\begin{aligned} d\phi_{p,h}(t) &= -\frac{h-l}{\sigma}\phi_{p,h}(t)dw(t) \\ d\phi_{p,l}(t) &= \frac{h-l}{\sigma}\phi_{p,l}(t)dw(t). \end{aligned}$$

Hence, $\phi_{p,h}$ and $\phi_{p,l}$ are geometric Brownian motions under probability measures P_h and P_l respectively and thus convenient to deal with. Finally, in the sequel $E^P[. \mid \mathcal{F}_t]$ denotes the conditional expectation with respect to beliefs P. We start the analysis by examining the benchmark case of a non-updating investor who never changes her initial beliefs.

[1] This is due to the fact that the unobservable drift is assumed to be constant.

1.2. Benchmark case: non-updating investor

1.2.1. Complete information

This case has been studied extensively in the literature (see for instance Dixit and Pindyck, chapter 6, p 180-185.). We briefly recall the main results. For $\mu \in \{l, h\}$, let β_μ be the positive root of the quadratic

$$\frac{\sigma^2}{2}x^2 + \left(\mu - \frac{\sigma^2}{2}\right)x - r = 0.$$

Notice that $\beta_\mu > 1$ since $r > \mu$. When μ is known and equal to l (respectively h), then $p = 0$ (respectively $p = 1$). The option value is given by

$$\begin{aligned} F^\mu(V) &= A_\mu V^{\beta_\mu} \text{ for } V \leq V_\mu^* \\ &= V - I \text{ for } V \geq V_\mu^* \ , \end{aligned}$$

with

$$V_\mu^* = \frac{\beta_\mu}{\beta_\mu - 1} I$$

and

$$A_\mu = \frac{1}{\beta_\mu}(V_\mu^*)^{1-\beta_\mu}.$$

We now look at the non-updating investor problem.

1.2.2. Non-updating investor problem

The investor chooses not to use arriving information: Changes in beliefs are simply equal to 0. At time 0, given the observation of the value of the project V and some initial beliefs p_0, an investor has to choose an optimal timing in order to maximize the benefits of investing, i.e.,

$$F_{NU}(V_0, p_0) = \sup_{\tau \geq 0} E^P\left[(V(\tau) - I)e^{-r(\tau - t)} \mid \mathcal{F}_0\right]$$

$$\begin{aligned} \text{s.t. } dV(s) &= V(s)\left((p_0 h + (1 - p_0)l)ds + \sigma d\overline{w}_{p_0}(s)\right) \\ dp(s) &= 0 \\ V(0) &= V_0 \text{ and } p(0) = p_0. \end{aligned}$$

To simplify notation, we drop the index 0 and write p instead of p_0 in the sequel.

For V remaining inside the inaction region, The Hamilton-Jacobi-Bellman (HJB) can be written

$$rF_{NU}(V,p)dt = E^P\left[dF_{NU}(V,p) \mid \mathcal{F}_0\right] \ .$$

Using Ito lemma leads to the following expression for the HJB

$$rF_{NU}(V,p) = V(ph + (1-p)l)F'_{NU}(V,p) + \frac{\sigma^2}{2}V^2 F''_{NU}(V,p).$$

This equation is similar to the one obtained under complete information when the average growth rate μ is known. The non-updating investor simply replaces the unknown value of μ by its average $ph + (1-p)l$ *once for good*. Using the initial condition $F_{NU}(0,p) = 0$, the solution of the HJB equation is

$$F_{NU}(V,p) = A(p)V^{\beta(p)},$$

where $\beta(p)$ is the positive root of the quadratic

$$\frac{\sigma^2}{2}x^2 + \left(ph + (1-p)l - \frac{\sigma^2}{2}\right)x - r = 0.$$

As in the complete information case, the optimal investment threshold $V_{NU}(p)$ is given by

$$V_{NU}(p) = \frac{\beta(p)}{\beta(p)-1}I,$$

and it follows that

$$F_{NU}(V,p) = (V_{NU}(p) - I)\left(\frac{V}{V_{NU}(p)}\right)^{\beta(p)} .$$

We now study the effect of initial optimism on the investment decision. We have the following proposition.

Proposition 1. *The more (initially) optimistic the investor is, the higher is the investment threshold.*

Proof.

See appendix 1. ■

The more optimistic, the higher the average return on the project $ph+(1-p)l$. The cost of the investment decreases by a factor e^{-rt} whereas on average the payoff of investing is reduced by $e^{-(r-(ph+(1-p)l))t}$. Optimism provides incentive to wait.

To conclude the analysis of the non-updating investor problem, we look at the average time until investment. Since p is maintained fixed, starting in state (V,p), the average time to investment T_{NU} is simply given by

$$\begin{aligned} T_{NU}(V,p) &= \frac{\ln\left(\frac{V_{NU}(p)}{V}\right)}{ph+(1-p)l-\frac{\sigma^2}{2}} \text{ for } ph+(1-p)l > \frac{\sigma^2}{2} \\ &= \infty \text{ otherwise.} \end{aligned} \tag{1.2}$$

Optimism has two opposing effects on the timing of investment. On the one hand, optimism increases the effective average growth rate of the project, which speeds up investment (direct effect). On the other hand, optimism raises the optimal investment threshold thus delaying investment (indirect effect). For $ph+(1-p)l > \frac{\sigma^2}{2}$, totalling differentiating relationship (1.2) with respect to p yields

$$\frac{\partial T_{NU}(V,p)}{\partial p} = \underbrace{\frac{1}{ph+(1-p)l-\frac{\sigma^2}{2}}\frac{1}{V_{NU}(p)}\frac{\partial V_{NU}(p)}{\partial p}}_{\text{indirect effect}} \underbrace{-(h-l)\frac{\ln\left(\frac{V_{NU}(p)}{V}\right)}{\left(ph+(1-p)l-\frac{\sigma^2}{2}\right)^2}}_{\text{direct effect}}.$$

We see that when V is small, the direct effect dominates and conversely, when V is close to $V_{NU}(p)$ the indirect effect dominates.

In the following section, we describe the Bayesian investor's program and compare the optimal investment frontier with the one obtained in the non-updating investor in order to determine the role of information in the irreversible investment decisions.

1.3. Bayesian investor's problem

At time $t=0$, the Bayesian investor's program is

$$F(V,p) = \sup_{\tau\geq 0} E^P\left[(V(\tau)-I)e^{-r(\tau-t)} \mid \mathcal{F}_t\right]$$

$$\begin{aligned}
\text{s.t. } dV(s) &= V(s)\left((p(s)h + (1-p(s))l)ds + \sigma d\overline{w}_p(s)\right) \\
dp(s) &= \frac{h-l}{\sigma} p(s)(1-p(s))d\overline{w}_p(s) \\
V(t) &= V \text{ and } p(t) = p.
\end{aligned}$$

Note that $\overline{w}_p$ is a standard Brownian motion under P. Details of the existence of the solution can be found in Øksendal (2000), Chapter 10. The supremum F is the least superharmonic majorant of the reward function $V - I$. In appendix 1 we prove that the inaction region IR of this problem can be written

$$IR = \{(t, V, p);\ 0 < V < V^*(p)\},$$

where V^* is the optimal investment frontier to be characterized in the sequel. Hence, for any (V, p) inside the inaction region IR, the Hamilton-Jacobi-Bellman (HJB) is

$$rF(V,p)dt = E^P\left[dF(V,p) \mid \mathcal{F}_t\right].$$

Using Ito lemma leads to the following expression for the HJB

$$\begin{aligned}
rF(V,p) &= V(ph + (1-p)l)F_1(V,p) + \frac{\sigma^2}{2}V^2F_{11}(V,p) \qquad (1.3) \\
&\quad + V(h-l)p(1-p)F_{12}(V,p) + \frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 (p(1-p))^2 F_{22}(V,p).
\end{aligned}$$

The initial condition is $F(0,p) = 0$ for all p since $V = 0$ is an absorbent state and the value-matching and smooth pasting (free boundary) conditions respectively are

$$\begin{aligned}
F(V^*(p), p) &= V^*(p) - I \\
\nabla F(V^*(p), p) &= (1, 0),
\end{aligned}$$

where $V^*(p)$ denotes the investment trigger value given the investor's beliefs p and $\nabla F = (F_1, F_2)$ is the gradient of F.

1.3.1. Interpretation of the value function

As usual, the return of investing an amount $F(V,p)$ into a safe asset with constant rate of return r must be equal to the expected capital gain from waiting (since no dividend is paid) governed by the changes in the value of the project and the beliefs. The two first terms on the right hand side of relationship (1.3) are the usual ones (given a fixed value for p) and represent the expected change in the option value as V varies. Appearing in the last term, $\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2(p(1-p))^2$ is a measure of informativeness, and $\frac{1}{r}F_{22}(V,p)$ is the shadow price of information. The gain from waiting is $\frac{1}{2r}\left(\frac{h-l}{\sigma}\right)^2(p(1-p))^2 F_{22}(V,p)$ and represents the direct effect of learning. In particular, if $h-l$ is small, σ is large or p is close to 0 or 1, the gain from waiting is small. On the contrary, the informativeness is maximal when $p=\frac{1}{2}$, i.e., when the investor is very confused about the true value of the drift μ. The median term $V(h-l)p(1-p)F_{12}(V,p)$ in relationship (1.3) represents the effects of change of beliefs on the marginal value of the option due to the correlation between the value of the project and the beliefs. The sign of the cross derivative F_{12} is somewhat difficult to predict. Nevertheless, when the drift μ is known, the marginal value of the option is decreasing in μ. When p increases, this somehow corresponds to a rise in the perceived value of the drift. This intuitive reasoning leads us to conjecture that F_{12} must be negative.

It appears that the magnitude of the uncertainty σ now plays an ambiguous role. On the one hand, an increase in σ rises the option value as in the classical case. On the other hand, when σ increases, less information can be extracted from the observations of V and therefore, it lowers the option value by decreasing the amount of information contained in the signal.

1.3.2. Properties of the option value and the investment trigger frontier

In this paragraph, we derive some useful properties about the option value F and the optimal investment trigger frontier V^*. The proofs are reported in appendix 2.

Property 1: F is strictly increasing and convex in its first argument. It follows that given p, $V^*(p)$ is uniquely defined.

Property 2: If at some date t, $p'(t)>p(t)$, then for all $s\geq t$, $p'(s)\geq p(s)$: If one investor is more optimistic than a second investor, she will always remain more optimistic.

Property 3: For all $V\geq 0$ and $p\in[0,1]$, $F(V,p)\leq pF(V,0)+(1-p)F(V,1)$.

Property 4: F is non decreasing and strictly convex in its second argument; Optimism increases the option value and information is always valuable.

Property 5: The optimal investment trigger frontier V^* is non decreasing in p. An optimistic investor requires a higher trigger value as she thinks that her option value of waiting is higher.

The key element to derive these properties is to use a change of probability (beliefs) using the Radon-Nikodym theorem (see for instance Cuoco and Zapatero (2000)) to rewrite the option value F as follows

$$\begin{aligned} F(V_0, p_0) &= \sup_{\tau \geq 0} \frac{1}{\xi_{p,l}(0)} E^l \left[\xi_{p,l}(\tau)(V(\tau) - I) e^{-r\tau} \mid \mathcal{F}_0 \right] \\ \text{s.t. } dV(t) &= V(t) \left(l dt + \sigma dw(t) \right) \\ \frac{\xi_{p,l}(t)}{\xi_{p,l}(0)} &= 1 - p_0 + p_0 \exp\left(-\frac{1}{2} \left(\frac{h-l}{\sigma} \right)^2 t + \frac{h-l}{\sigma} w(t) \right). \end{aligned}$$

In the next paragraph, we compare the investment strategy of a Bayesian investor who uses the arrival of information to update her beliefs with the investment strategy of a non-updating investor whose beliefs do not evolve over the course of time.

1.3.3. Bayesian investor versus non-updating investor

The non-updating investor optimal strategy is used as a benchmark to study the role of information in investment decision. We have the following proposition.

Proposition 2. *The investment trigger of the Bayesian-investor is higher than the investment trigger of the non-updating investor, i.e.,*

$$\begin{aligned} V^*(p) &< V_{NU}(p) \text{ for all } p \text{ in } (0,1)\text{, and} \\ V^*(0) &= V_{NU}(0) \text{ and } V^*(1) = V_{NU}(1). \end{aligned}$$

In particular, this gives us an upper bound for the optimal investment frontier of the Bayesian firm.

Proof.

See appendix 3. ■

Proposition 3. states that the optimal investment of a non-updating investor lies above the Bayesian investor's one. One natural concern is then: is there any monotonic relationship between the two option values? Unfortunately, the answer is negative. Numerical simulations (not displayed here) show that when the value of the project V is small, the Bayesian investor has a higher option value than the non-updating investor for any given p; the opposite result holds when V is close to the investment trigger of the Bayesian firm. The non-updating investor gives up the opportunity to learn about the characteristics of the project. In compensation, she requires a higher premium over the cost of investing to undertake an irreversible commitment. In some sense, this result reinforces Bernanke's bad news principle. Moreover, this result is the mirror of the one obtained by Venezia (1983) who considers a firm having an option (put) to sell an asset and shows that the Bayesian firm has more incentive to wait than the non-updating firm.

1.4. Average time until investment

In this paragraph, we aim at studying the effects of optimism on the timing of investment. On the one hand, we have already seen that optimistic agents require a higher investment trigger value since they think their option is higher. On the other hand, a higher p increases the (perceived) value of the growth rate of the project, which enhances the travelling speed of the stochastic process V accelerating the decision of investing. Hence, we are interested in answering the following question: Do optimistic agents tend to rush or to delay investment on average? Given the initial value of a couple (V, p), let $T(V, p)$ denote the average time a Bayesian investor may expect to wait until investment. In order to ensure existence of T for all p in $[0, 1]$, the condition $l - \frac{\sigma^2}{2} > 0$ is needed. We now characterize the average time T.

Proposition 3. *The average time T until investment satisfies the following Hamilton Jacobi Bellman equation*

$$\begin{aligned} -1 &= V(ph + (1-p)l)T_1(V,p) + \frac{\sigma^2}{2}V^2 T_{11}(V,p) \\ &\quad + V(h-l)p(1-p)T_{12}(V,p) + \frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 (p(1-p))^2 T_{22}(V,p), \end{aligned}$$

with the initial condition

$$\lim_{V\to 0} T(V,p) = \infty, \text{ for all } p \in [0,1],$$

and boundary condition

$$T(V^*(p), p) = 0 \text{ for all } p \in [0,1].$$

In addition,

$$T(V,1) = \frac{\ln\left(\frac{V^*(1)}{V}\right)}{h - \frac{\sigma^2}{2}} \text{ and } T(V,0) = \frac{\ln\left(\frac{V^*(0)}{V}\right)}{l - \frac{\sigma^2}{2}}.$$

Proof. The central idea of the proof relies on the construction of an appropriate martingale in order to use the *Optional Stopping Theorem*. A complete proof is provided in appendix 4. ■

Another interesting issue is: How long on average does a Bayesian investor wait until investing when the project is bad, i.e., $\mu = l$ (respectively good, $\mu = h$)? Using the law of conditional expectations we have

$$E[\tau^*] = pE[\tau^* \mid \mu = h] + (1-p)E[\tau^* \mid \mu = l].$$

Given the fact that $\mu \in \{l, h\}$, the law of motion of V is

$$dV(t) = V(t)(\mu dt + \sigma dw(t)),$$

and the law of motion of the beliefs p is

$$dp(t) = \frac{h-l}{\sigma} p(t)(1-p(t)) \left(\varepsilon \frac{h-l}{\sigma} p^{\frac{1-\varepsilon}{2}}(t)(1-p(t))^{\frac{1+\varepsilon}{2}} + dw(t)\right),$$

where $\varepsilon = 1$ if $\mu = h$ and $\varepsilon = -1$ if $\mu = l$. Note that the beliefs p is a submartingale (supermartingale) if $\mu = l$ ($\mu = h$), reflecting the fact that the Bayesian investor's beliefs decrease (increase) if $\mu = l$ ($\mu = h$). Once again, using the optional sampling theorem, it is easy to show that for $\mu \in \{l, h\}$, given the initial condition (V, p), the average time T^μ until investment satisfies the following Hamilton Jacobi Bellman equation

$$\begin{aligned} -1 &= \mu V T_1^\mu(V,p) + \frac{\sigma^2}{2} V^2 T_{11}^\mu(V,p) + \varepsilon \left(\frac{h-l}{\sigma}\right)^2 p(1-p)p^{\frac{1-\varepsilon}{2}}(1-p)^{\frac{1+\varepsilon}{2}} T_2^\mu \\ &\quad + V(h-l)p(1-p)T_{12}^\mu(V,p) + \frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 (p(1-p))^2 T_{22}^\mu(V,p), \end{aligned}$$

with the initial condition

$$\lim_{V \to 0} T^{\mu}(V, p) = \infty, \text{ for all } p \in [0, 1],$$

and boundary condition

$$T^{\mu}(V^*(p), p) = 0 \text{ for all } p \in [0, 1].$$

In general, it is not possible to get a closed form solution for the option value and the average time until investment. The reason is that it requires us to solve a second order parabolic partial differential equation with a free boundary condition. Hence, we have to rely on numerical methods. However, it turns out that for the special case where $l + h = \sigma^2$, closed forms solutions can be obtained. This special case is presented in the next session.

2. A Tractable Case: $h + l = \sigma^2$

Although the condition $h + l = \sigma^2$ is restrictive and in particular prevents us from doing some comparative statics over σ, it allows us to explicitly determine the optimal investment frontier and provide a representation for the option value. We start with the following proposition.

Proposition 4. *The option value is given by*

$$\begin{aligned} F(V, p) &= (1-p)A\left(\frac{p}{1-p}V^{-\frac{h-l}{\sigma^2}}\right)V^{\beta_l} \text{ for } V \leq V^*(p) \\ &= V - I \text{ for } V \geq V^*(p), \end{aligned}$$

and the optimal investment frontier is given by

$$V^*(p) = \left(1 + \frac{1}{p\beta_h + (1-p)\beta_l - 1}\right)I.$$

In addition, the function A is defined on $\mathbb{R}_{++}$ and is given by

$$A(x) = \frac{1}{1 - S(x)}\left(\left(\frac{x(1 - S(x))}{S(x)}\right)^{-\frac{\sigma^2}{h-l}} - I\right)\left(\frac{x(1 - S(x))}{S(x)}\right)^{\frac{\sigma^2 \beta_l}{h-l}}$$

$$\lim_{x \longrightarrow 0} A(x) = A_l$$
$$\lim_{x \longrightarrow \infty} \frac{A(x)}{x} = A_h,$$

where the function S is defined on $\mathbb{R}_{++}$ into $[0,1]$ as the inverse function of

$$p \longmapsto \frac{p}{1-p}(V^*(p))^{-\frac{h-l}{\sigma^2}}.$$

Proof.

See appendix 5. ■

The closed form solution obtained for the optimal investment frontier allows us to draw a direct comparison between the Bayesian and non-updating investor optimal strategies. Recall that in the non-updating investor's case, the optimal investment frontier is given by

$$V_{NU}(p) = \left(1 + \frac{1}{\beta(p) - 1}\right) I.$$

Since $\beta(.)$ is a convex function (see appendix 1), it follows that $\beta(p) \leq p\beta_h + (1-p)\beta_l$ and therefore we have

$$V^*(p) \leq V_{NU}(p).$$

Simple algebra also reveals that V^* is convex in p so the more optimistic the investor is getting the more she wants to wait. We conclude this section by examining the average time until investment which is only defined for p in $\left[\frac{1}{2}, 1\right]$.

Proposition 5. *Starting at (V,p) in the inaction region IR and $p \geq \frac{1}{2}$, the average time T until investment is given by*

$$T(V,p) = \left(\frac{1-p}{l - \frac{\sigma^2}{2}} + \frac{p}{h - \frac{\sigma^2}{2}}\right) \ln\left(\frac{V^*(S(\frac{p}{1-p}V^{-\frac{h-l}{\sigma^2}})}{V}\right),$$

Proof.

See appendix 5. ■

In appendix 5, we also show that

$$p \longmapsto \frac{p}{1-p}(V^*(p))^{-\frac{h-l}{\sigma^2}},$$

is a strictly increasing function, which implies that its inverse S is also increasing. Moreover, the functions $p \longmapsto \frac{p}{1-p}$ and V^* are also increasing. It follows that given V, T is an increasing function of p. Optimism in this case unambiguously delays investment.

The next section is devoted to some numerical simulations showing the impact of uncertainty and optimism on the optimal investment frontier and the average time until investment.

3. Numerical Simulations

We start this section by presenting the methodology used for the numerical simulations. We actually use two distinct methods.

3.1. Method 1: binomial tree

The first method and easier to implement is a binomial tree approach. It relies on the following representation for the option value

$$F(V_0, p_0) = \sup_{\tau \geq 0} E^l \left[\left(1 - p_0 + p_0 \left(\frac{V(\tau)}{V} \right)^{\frac{h-l}{\sigma^2}} e^{-a\tau} \right) (V(\tau) - I) e^{-r\tau} \mid \mathcal{F}_0 \right],$$

where $a = \frac{h-l}{\sigma^2}\left(\frac{h+l}{2} - \frac{\sigma^2}{2}\right) > 0$ and $V(t) = V_0 \exp\left((l - \frac{\sigma^2}{2})t + \sigma w(t)\right)$. We approximate the infinite horizon call option with a finite horizon call option with terminal date T. Then, for N a positive integer, one period length is $\Delta t = \frac{T}{N}$. The process V can be approximated by a Bernoulli process such that, given $V(t)$, we have

$$V(t + \Delta t) = \begin{cases} uV(t) \text{ with probability } q \\ dV(t) \text{ with probability } 1 - q \end{cases}$$

with $u = e^{\sigma\sqrt{\Delta t}}$, $d = u^{-1}$ and $q = \frac{e^{l\sqrt{\Delta t}} - d}{u - d}$.

Beliefs p_0 only appears as a parameter. This method only works for computing the option value F and the optimal trigger investment frontier V^*. This latter is obtained by a bisection method. Given p_0, we know that $V^*(p_0)$ is in the interval $[V^*(0), V^*(1)]$. We compute the option value for the midpoint $V_m^1 = \frac{1}{2}(V^*(0) + V^*(1))$, i.e., $F(V_m^1, p_0)$. If $F(V_m^1, p_0) > V_m^1 - I$, $V^*(p_0)$ must be in $[V^*(0), V_m^1]$, and we continue the procedure with a new midpoint $V_m^2 = \frac{1}{2}(V^*(0) + V_m^1)$. If $F(V_m^1, p_0) < V_m^1 - I$, $V^*(p_0)$ must be in $[V_m^1, V^*(1)]$, and we continue the procedure with a new midpoint $V_m^2 = \frac{1}{2}(V_m^1 + V^*(1))$. For an arbitrary small tolerance $\varepsilon > 0$, we stop the algorithm after N iterations when $\left|V_m^N - V_m^{N-1}\right| \leq \varepsilon$. The second method is a finite difference approach, harder to implement but it provides much more flexibility and in particular can be applied to compute the expected time to investment.

3.2. Method 2: finite differences

For (V, p) in $[0, V^*(1)] \times [0, 1]$, we discretize the HJB equation choosing a $N_V \times N_p$ grid. Partial derivatives are approximated by (central) finite difference equations. A particular point of the grid is (V, p) with $V = i\Delta V$ and $p = j\Delta p$ for $(i, j) \in [1, N_V] \times [1, N_p]$. Then, by re-indexation $k = (i-1) \times N_p + j$, we convert the problem into solving a $N = N_V \times N_p$ linear system of the type

$$AF = B,$$

where A is a $N \times N$ square matrix, B is a $N \times 1$ vector incorporating the boundary conditions $F(0, p)$, $F(V, 0)$, $F(V, 1)$ and $F(V^*(1), p) = V^*(1) - I$. The free boundary condition is dealt with by using successive over-relaxations (SOR), which means that at each iteration, we check that the value obtained for the option value is above the corresponding payoff of exerting the option. If not, the computed value is replaced by the corresponding payoff. One drawback with this method is that it requires to solve a linear system whose size grows very quickly with the degree of precision desired. The main advantage is that we obtain *all* the values for F and can be implemented for elliptic systems, i.e. when dealing with a two dimensional Wiener process.

Tables 1 is obtained using the binomial method and we get an acceptable precision for $T = 50$ and $N = 10,000$ if we compare the computed values and the exact values of the investment trigger when $p = 0$ and $p = 1$. It displays the value of the optimal investment frontier for both a Bayesian and non-updating investors for different values of the beliefs p and uncertainty σ. We also compute $\Delta = \frac{V_{NU} - V^*}{V^*}$ the relative difference in investment triggers between the two types of investors for some given beliefs p.

TABLE 1
Optimal trigger investment frontiers

σ^2		0.1			0.3			0.8	
	V_{NU}	V^*	Δ	V_{NU}	V^*	Δ	V_{NU}	V^*	Δ
p									
0	1.564	1.564	0	1.871	1.871	0	2.457	2.457	0
0.1	1.654	1.604	0.031	1.972	1.923	0.025	2.591	2.572	0.007
0.2	1.761	1.667	0.056	2.091	2.012	0.040	2.748	2.706	0.015
0.3	1.888	1.748	0.080	2.237	2.116	0.055	2.931	2.865	0.023
0.4	2.041	1.851	0.102	2.402	2.247	0.069	3.15	3.057	0.030
0.5	2.227	2.001	0.113	2.608	2.404	0.085	3.414	3.293	0.037
0.6	2.456	2.209	0.112	2.863	2.614	0.095	3.739	3.590	0.042
0.7	2.746	2.474	0.110	3.183	2.902	0.097	4.147	3.975	0.043
0.8	3.120	2.820	0.107	3.600	3.300	0.093	4.675	4.495	0.040
0.9	3.621	3.377	0.072	4.156	3.890	0.068	5.380	5.235	0.028
1	4.325	4.325	0	4.940	4.940	0	6.372	6.372	0

$r = 0.8,\ h = 0.6,\ l = 0.2,\ I = 1.$

Table 1 shows that both V^* and V_{NU} are convex in p. When the investor is fairly optimistic, a change in beliefs has a high impact on the optimal trigger investment value. In addition, an increase in uncertainty σ delays investment: For all values of the beliefs p, both V^* and V_{NU} rise and the relative difference Δ between the two thresholds shrinks. When σ is large, the signal is not very informative and the investor has a more cautious attitude: She waits more and for any given beliefs p, her optimal trigger investment value is closer to the one of the non-updating investor. The relative difference Δ measures the effect of learning on the irreversible investment decision. We see that the effect is maximum for p within the range $[0.5, 0.7]$ - this is due to the convexity of the optimal investment frontiers -, and rather small. Regarding the last point, remember that the payoffs of the option only depend on the value of the project and therefore, the impact of the beliefs is of second order.

TABLE 2

Average times until investment T^*, T_{NU}

$V = 0.49$

p	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
T^*	26.83	25.16	23.31	21.33	19.25	16.81	14.30	12.04	9.75	7.43	5.14
T_{NU}	26.83	15.49	11.18	8.93	7.58	6.70	6.09	5.68	5.40	5.22	5.14

$V = 1.71$

p	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
T^*	1.77	2.35	2.86	3.23	3.55	3.74	3.80	3.69	3.42	3.00	2.35
T_{NU}	1.77	1.57	1.54	1.56	1.61	1.68	1.77	1.88	2.01	2.16	2.35

$V = 2.83$

p	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
T^*	0	0	0	0	0	0	0	0.27	0.82	1.17	1.23
T_{NU}	0	0	0	0	0	0	0.03	0.35	0.64	0.93	1.23

$r = 0.8,\ h = 0.6,\ l = 0.2,\ \sigma^2 = 0.3,\ I = 1$

Table 2 reveals that optimism has different effects on the non-updating and the Bayesian investors' average times until investment. When V is small, $T_{NU}(V,.)$ is decreasing, then becomes U shape as V rises and finally when V is high enough $T_{NU}(V,.)$ is non-decreasing, possibly flat (and equal to zero) for small values of p When V is small, $T^*(V,.)$ is decreasing, then becomes hump shape as V rises and ultimately when V is high enough $T^*(V,.)$ is non-decreasing, possibly flat (and equal to zero) for small values of p. We notice that unless V is very close to the investment trigger frontier, the average until investment of the Bayesian investor is greater than the one of the non-updating investor.

4. Conclusion

We used a very simple model based on the work of McDonald and Siegel (1986) to explore the implications of incomplete information on irreversible investment decisions. Such a framework applies to many real life investment opportunities. Observing the realizations of the project over time provides some information about the true value of the average growth rate of the project and then gives incentives to wait to learn about the true characteristics of the project. The role of uncertainty is twofold. On the one hand, more uncertainty raises the option value. On the other hand, more uncertainty reduces the quality of the information received. Optimistic agents have a higher option value to wait and therefore, choose to postpone their investment. Taking as a benchmark an investor who does not update her beliefs, we prove that using information accelerates the decision to invest measured in terms of investment threshold. An investor who does not use information requires a higher premium over the cost of investing in order to accept an irreversible investment commitment. Finally, numerical simulations show that beliefs have different effects on the average time until investment of the Bayesian and non-updating investors. Unless, the value of the project is very close to the investment trigger frontier, the average time of the Bayesian investor is greater than the one of the non-updating investor. A possible extension of the paper will be to allow the investor to have access to an additional and costly signal. This is left for future research.

5. Appendix

5.1. Appendix 1

Inaction Region. We want to show that given p_0, if V_0 is in IR, then $W_0 < V_0$ is also in IR. Assume V_0 is in IR, then $F(V_0, p_0) > V_0 - I$. Let τ_{V_0} be the optimal stopping time when the process V starts at V_0. Writing $V(t) = V_0 K(t)$ with $K(t) = \exp\left(-\int_0^t \left(p(s)h + (1-p(s))l - \frac{1}{2}\sigma^2\right) ds + \sigma \overline{w}_p(t)\right)$, it follows that

$$
\begin{aligned}
F(W_0, p_0) &= \sup_{\tau \geq 0} E^P\left[(W_0 K(\tau) - I)e^{-r\tau} \mid \mathcal{F}_0\right] \\
&\geq E^P\left[(W_0 K(\tau_{V_0}) - I)e^{-r\tau_{V_0}} \mid \mathcal{F}_0\right] \\
&\geq E^P\left[\frac{W_0}{V_0}(V_0 K(\tau_{V_0}) - I)e^{-r\tau_{V_0}} + \frac{W_0 - V_0}{V_0} I e^{-r\tau_{V_0}} \mid \mathcal{F}_0\right] \\
&\geq \frac{W_0}{V_0} F(V_0, p_0) + \frac{W_0 - V_0}{V_0} E^P\left[I e^{-r\tau_{V_0}} \mid \mathcal{F}_0\right] \\
&\geq \frac{W_0}{V_0} F(V_0, p_0) + \frac{W_0 - V_0}{V_0} I \text{ (since } W_0 < V_0 \text{ and } E^P\left[I e^{-r\tau_{V_0}} \mid \mathcal{F}_0\right] \leq I) \\
&> \frac{W_0}{V_0}(V_0 - I) + \frac{W_0 - V_0}{V_0} I \\
&> W_0 - I, \text{ which shows that } W_0 \text{ is in } IR.
\end{aligned}
$$

Proof of Proposition 1. Recall that $V_{NU}(p) = \frac{\beta(p)}{\beta(p)-1} I$. Hence, to prove that $V_{NU}(p)$ is increasing in p, it is enough to show that $\beta(p)$ is decreasing in p. By definition, $\beta(p)$ satisfies

$$
\frac{\sigma^2}{2}\beta^2(p) + \left(ph + (1-p)l - \frac{\sigma^2}{2}\right)\beta(p) - r = 0.
$$

By using the Implicit Function Theorem, it is easy to show that $\beta(.)$ is a differentiable function in p and totally differentiating the previous equation yields

$$
\sigma^2 \beta(p) \frac{d\beta(p)}{dp} + \left(ph + (1-p)l - \frac{\sigma^2}{2}\right)\frac{d\beta(p)}{dp} + (h-l)\beta(p) = 0.
$$

It follows

$$
\frac{d\beta(p)}{dp}\left(r + \frac{\sigma^2}{2}\beta^2(p)\right) = -(h-l)\beta^2(p).
$$

We readily conclude that $\frac{d\beta(p)}{dp} < 0$ and the desired result follows. Moreover, differentiating again with respect to p the previous relationship yields

$$\sigma^2 \frac{d\beta(p)}{dp} + \frac{d^2\beta(p)}{dp^2}\left(r + \frac{\sigma^2}{2}\beta^2(p)\right) = -2(h-l)\beta(p)\frac{d\beta(p)}{dp},$$

so

$$\frac{d^2\beta(p)}{dp^2}\left(r + \frac{\sigma^2}{2}\beta^2(p)\right) = -\left(2(h-l)\beta(p) + \sigma^2\right)\frac{d\beta(p)}{dp}.$$

We conclude that $\frac{d^2\beta(p)}{dp^2} > 0$ or $\beta(.)$ is strictly convex in p. ■

5.2. Appendix 2

Rewriting the option value F under the probability measure P_l by using the Radon-Nikodym derivative theorem leads to

$$F(V,p) = \sup_{\tau \geq 0} \frac{1}{\xi_{p,l}(0)} E^l\left[\xi_{p,l}(\tau)(V(\tau) - I)e^{-r\tau} \mid \mathcal{F}_0\right].$$

We know that $\xi_{p,l}(t) = \frac{1-p}{1-p(t)}$ with $p = p(0)$. Let us define $\phi_{p,l}(t) = \frac{p(t)}{1-p(t)}$ and using Ito lemma leads to

$$d\phi_{p,l}(t) = \frac{h-l}{\sigma}\phi_{p,l}(t)dw(t).$$

Hence

$$\phi_{p,l}(t) = \phi_{p,l}(0)\exp\left(-\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 t + \frac{h-l}{\sigma}w(t)\right).$$

Thus

$$\frac{\xi_{p,l}(t)}{\xi_{p,l}(0)} = 1 - p + p\exp\left(-\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 t + \frac{h-l}{\sigma}w(t)\right).$$

To save notations, we set $\kappa(t) = \exp\left(-\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 t + \frac{h-l}{\sigma}w(t)\right)$. Finally,

$$F(V,p) = \sup_{\tau \geq 0} E^l\left[(1 - p + p\kappa(\tau))\,(V(\tau) - I)e^{-r\tau} \mid \mathcal{F}_0\right]$$

Proof of Property 1. Writing

$$F(V_0,p_0) = \sup_{\tau \geq 0} E^l\left[(1 - p_0 + p_0\kappa(\tau))\,(V_0 e^{(l-\frac{\sigma^2}{2})\tau + w(\tau)} - I)e^{-r\tau} \mid \mathcal{F}_0\right]$$

readily shows that F is strictly increasing in V inside the inaction region. Then, let λ be in $(0,1)$ and V_1 and V_2 two geometric Brownian motions with the same law of motion under P_l. For p_0 in $[0,1]$, let $\overline{V}_0 = \lambda V_{10} + (1-\lambda)V_{20}$, then we have

$$\begin{aligned}
F(\overline{V}_0, p_0) &= \sup_{\tau\geq 0} E^P\left[(\lambda V_1(\tau) + (1-\lambda)V_2(\tau) - I)e^{-r\tau} \mid \mathcal{F}_0\right] \\
&= \sup_{\tau\geq 0} E^l\left[(1-p_0+p_0\kappa(\tau))\left((\lambda V_1(\tau)+(1-\lambda)V_2(\tau)-I)-I\right)e^{-r\tau} \mid \mathcal{F}_0\right] \\
&\leq \lambda\sup_{\tau\geq 0}\left[E^l\left(1-p_0+p_0\kappa(\tau)\right)(V_1(\tau)-I)e^{-r\tau} \mid \mathcal{F}_0\right] + \\
&\quad (1-\lambda)\sup_{\tau\geq 0} E^l\left[\left(1-p_0+p_0\kappa(\tau)\right)(V_2(\tau)-I)e^{-r\tau} \mid \mathcal{F}_0\right] \\
&\leq \lambda F(V_{10}, p_0) + (1-\lambda)F(V_{20}, p_0).
\end{aligned}$$

Unless we are outside the inaction region (in this case, the optimal stopping time is zero), the inequality is actually a strict inequality since the optimal stopping times for project 1 and for project 2 are distinct as V_1 and V_2 are distinct inside the inaction region so F is *strictly* increasing in V. For any given p in $[0,1]$, the strict convexity in V ensures that $V^*(p)$ is unique. ■

Proof of Property 2. Let p and p' be two *one-dimensional Markovian processes* following the *same* law of motion. If at some date θ, $p'(\theta) = p(\theta)$, then we have $p' = p$ for all dates $s \geq \theta$. It follows that if $p'(0) > p(0)$ then $p' \geq p$ for all $t \geq \tau$. ■

Proof of Property 3.

$$\begin{aligned}
F(V,p) &= \sup_{\tau\geq 0} E^P\left[(V(\tau)-I)e^{-r\tau} \mid \mathcal{F}_0\right] \\
&= \sup_{\tau\geq 0}\{pE^h\left[(V(\tau)-I)e^{-r\tau} \mid \mathcal{F}_0\right] + (1-p)E^l\left[(V(\tau)-I)e^{-r\tau} \mid \mathcal{F}_0\right]\} \\
&\leq p\sup_{\tau\geq 0} E^h\left[(V(\tau)-I)e^{-r\tau} \mid \mathcal{F}_0\right] + (1-p)\sup_{\tau\geq 0} E^l\left[(V(\tau)-I)e^{-r\tau} \mid \mathcal{F}_0\right] \\
&\leq pF(V,1) + (1-p)F(V,0). \blacksquare
\end{aligned}$$

Proof of Property 4.

Step 1: For all $V \geq 0$, F is non-decreasing in p.

Proof. For any $V \geq 0$, we want to show that if $p' \geq p$, then $F(V,p') \geq F(V,p)$. First, we prove the following lemma.

Lemma 2. *If there are two projects V_1 and V_2 such that at time 0, $V_1(0) = V_2(0)$*

whose laws of motion are given by

$$\begin{aligned} dV_1(t) &= V_1(t)\left(\mu_1(t)dt + \sigma dw(t)\right) \\ dV_2(t) &= V_2(t)\left(\mu_2(t)dt + \sigma dw(t)\right), \end{aligned}$$

where $\mu_2 \geq \mu_1$*, then the option value* G_2 *associated with project* 2 *is higher than the option value* G_1 *associated with project* 1*.*

Proof. Define a new process $X = \frac{V_2}{V_1}$ and $X(0) = 1$. Using Ito lemma yields

$$dX(t) = X(t)\left(\mu_2(t) - \mu_1(t)\right)dt.$$

X is therefore a deterministic process with a non-negative mean and starting at 1. Thus, for all time $t \geq 0$, $X(t) \geq 1$. We conclude that $V_2 \geq V_1$. By definition,

$$G_2(V) = \sup_{\tau \geq 0} E\left[(V_2(\tau) - I)e^{-r\tau} \mid \mathcal{F}_0\right].$$

Since $V_2 \geq V_1$, in particular, for all stopping time τ, we have

$$E\left[(V_2(\tau) - I)e^{-r\tau} \mid \mathcal{F}_0\right] \geq E\left[(V_1(\tau) - I)e^{-r\tau} \mid \mathcal{F}_0\right].$$

Hence,

$$G_2(V) \geq G_1(V). \blacksquare$$

Now, recall that

$$F(V,p) = \sup_{\tau \geq 0} E^P\left[(V(\tau) - I)e^{-r\tau} \mid \mathcal{F}_0\right]$$

$$\begin{aligned} \text{s.t. } dV(t) &= V(t)\left((p(t)h + (1-p(t))l)dt + \sigma d\overline{w}_p(t)\right) \\ dp(t) &= \frac{h-l}{\sigma} p(t)(1-p(t))d\overline{w}_p(t). \end{aligned}$$

The average growth rate of the project is $ph + (1-p)l$. Also recall by property 1 that if at some date the beliefs p' is greater than the beliefs p, then it will remain greater than p at all subsequent dates. The desired result follows straightforwardly from the previous lemma.

Step 2: For all $V \geq 0$, F is strictly convex in p.

Let $\lambda \in (0,1)$, $(p,p') \in [0,1]^2$ and $p'' = \lambda p + (1-\lambda)p'$. Then,

$$\begin{aligned}
F(V,p'') &= \sup_{\tau \geq 0} E^l \left[(1 - p'' + p''\kappa(\tau)) (V(\tau) - I)e^{-r\tau} \mid \mathcal{F}_0 \right] \\
&= \sup_{\tau \geq 0} E^l \left[(\lambda(1 - p + p\kappa(\tau)) + (1-\lambda)(1 - p' + p'\kappa(\tau))) (V(\tau) - I)e^{-r\tau} \mid \mathcal{F}_0 \right] \\
&= \sup_{\tau \geq 0} \left\{ \lambda \frac{1}{\xi_{p,l}(0)} E^l \left[\xi_{p,l}(\tau)(V(\tau) - I)e^{-r\tau} \mid \mathcal{F}_0 \right] \right. \\
&\quad \left. + (1-\lambda) \frac{1}{\xi_{p',l}(0)} E^l \left[\xi_{p',l}(\tau)(V(\tau) - I)e^{-r\tau} \mid \mathcal{F}_0 \right] \right\} \\
&< \lambda F(V,p) + (1-\lambda)F(V,p'),
\end{aligned}$$

which shows that F is strictly convex in p. ■

Proof of Property 5. Since for any $V > 0$ and $p' \geq p$, $F(V,p') \geq F(V,p)$, using the value matching condition, we have $F(V^*(p),p') \geq V^*(p) - I$. Thus, it follows easily that for $p' \geq p$, $V^*(p') \geq V^*(p)$. ■

5.3. Appendix 3

Proof of Proposition 2. We want show that, for p in $(0,1)$ when the Bayesian firm is about to invest (τ close to 0), the value of waiting of the non-updating firm is strictly greater. Recall that

$$F(V,p) = \sup_{\tau \geq 0} E^l \left[(1 - p + p\kappa(\tau)) (V(\tau) - I)e^{-r\tau} \mid \mathcal{F}_0 \right],$$

where $\kappa(t) = \exp\left(-\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 t + \frac{h-l}{\sigma} w(t)\right)$. Since $\mu = l$, it is easy to see that F can be written

$$F(V,p) = \sup_{\tau \geq 0} E^l \left[\left(1 - p + p\left(\frac{V(\tau)}{V}\right)^{\frac{h-l}{\sigma^2}} e^{-a\tau}\right) (V(\tau) - I)e^{-r\tau} \mid \mathcal{F}_0 \right],$$

where $a = \frac{h-l}{\sigma^2}\left(\frac{h+l}{2} - \frac{\sigma^2}{2}\right)$. Conversely, in the case of a non-updating investor, p is a constant and therefore the density process $\xi_{p,l}$ (Radon-Nikodym derivative with respect to P_l) is equal to

$$\xi_{p,l}(t) = \exp\left(-\frac{1}{2}p^2\left(\frac{h-l}{\sigma}\right)^2 t + p\frac{h-l}{\sigma} w(t)\right).$$

Hence, the option value F_{NU} of a non-updating investor is

$$F_{NU}(V,p) = \sup_{\tau \geq 0} E^l \left[(\xi_{p,l}(\tau) \left(V(\tau) - I \right) e^{-r\tau} \mid \mathcal{F}_0 \right].$$

Again, since $\mu = l$, F_{NU} can be written

$$F_{NU}(V,p) = \sup_{\tau \geq 0} E^l \left[\left(\left(\frac{V(\tau)}{V} \right)^{p\frac{h-l}{\sigma^2}} e^{-p\frac{h-l}{2\sigma^2}\left(p(h+l)-\sigma^2\right)\tau} \right) (V(\tau) - I) e^{-r\tau} \mid \mathcal{F}_0 \right].$$

We know that no investment will be made for both firms as long as $V(\tau) \leq I$. In order to show that for p in $(0,1)$, $F_{NU}(V,p) > F(V,p)$ when the Bayesian firm is about to invest, it is enough to that for $t > 0$ and $(t, V(t))$ in the neighborhood of $(0, V)$, we have $\left(\frac{V(t)}{V}\right)^{p\frac{h-l}{\sigma^2}} e^{-p\frac{h-l}{2\sigma^2}\left(p(h+l)-\sigma^2\right)t} > \left(1 - p + p\left(\frac{V(\tau)}{V}\right)^{\frac{h-l}{\sigma^2}} e^{-at}\right)$. Let us set $\left(\frac{V(t)}{V}\right)^{\frac{h-l}{\sigma^2}} = e^y$ and therefore we want to show that some t and y small enough

$$e^{py} e^{-p\frac{h-l}{2\sigma^2}\left(p(h+l)-\sigma^2\right)t} > 1 - p + pe^{y-at}.$$

Notice that $e^{py} e^{-p\frac{h-l}{2\sigma^2}\left(p(h+l)-\sigma^2\right)t} = e^{p(y-at)} e^{p(1-p)\frac{h-l}{2\sigma^2}(l+l)t}$. Thus the first order Taylor expansion of the LHS around $(0,0)$ is equal to

$$LHS(y,t) = 1 + p(y - at) + p(1-p)\frac{h-l}{2\sigma^2}(h+l)\,t + o(y,t).$$

In the same way, the first order Taylor expansion of the LHS around $(0,0)$ is equal to

$$RHS(y,t) = 1 + p(y - at) + o(y,t).$$

Since for p in $(0,1)$, $p(1-p)\frac{h-l}{2\sigma^2}(h+l) > 0$, we have

$$LHS(y,t) > RHS(y,t),$$

for $t > 0$ and $(t, V(t))$ in the neighborhood of $(0, V)$. It follows that if τ^* is the optimal stopping time for the Bayesian firm and *small enough, because Ito processes have continuous paths,* we have

$$\begin{aligned} & E^l \left[\left(1 - p + p\left(\frac{V(\tau^*)}{V}\right)^{\frac{h-l}{\sigma^2}} e^{-a\tau^*} \right) (V(\tau^*) - I) e^{-r\tau^*} \mid \mathcal{F}_0 \right] \\ < \; & E^l \left[\left(\left(\frac{V(\tau^*)}{V}\right)^{p\frac{h-l}{\sigma^2}} e^{-p\frac{h-l}{2\sigma^2}\left(p(h+l)-\sigma^2\right)\tau^*} \right) (V(\tau^*) - I) e^{-r\tau^*} \mid \mathcal{F}_0 \right]. \end{aligned}$$

Since τ^* is one possible stopping time for the non-updating firm and may not be optimal, it follows that

$$E^l \left[\left(\left(\frac{V(\tau^*)}{V} \right)^{p\frac{h-l}{\sigma^2}} e^{-p\frac{h-l}{2\sigma^2}\left(p(h+l)-\sigma^2\right)\tau^*} \right) (V(\tau^*) - I)e^{-r\tau^*} \mid \mathcal{F}_0 \right] \leq F_{NU}(V, p).$$

Hence, in the neighborhood of $(0, V)$, we have

$$F(V, p) < F_{NU}(V, p).$$

It follows that for p in $(0, 1)$, we must have

$$V^*(p) < V_{NU}(p),$$

with

$$V^*(0) = V_{NU}(0) \text{ and } V^*(1) = V_{NU}(1). \blacksquare$$

5.4. Appendix 4

Proof of Proposition 3. We define

$$\tau^* = T(V, p) = \inf\{t \geq 0 : V(t) = V^*(p(t)) \mid V(0) = V,\ p(0) = p\}.$$

We assume that $P(\tau^* < \infty) = 1$. Clearly, we have $T(V^*(p), p) = 0$ and $T(V, p) = 0$ for $V > V^*(p)$. One way to compute the expected time is to determine a suitable martingale $\mathcal{M}$ and exploit the following martingale property: $E^P\left[\mathcal{M}(\tau)\right] = \mathcal{M}(0)$. This is the central idea of the **Optional Stopping Theorem**. First, we briefly check that the conditions to use the Optional Stopping Theorem are met. We have shown that for all p in $[0, 1]$, $V^*(p) \leq V^*(1)$. Hence (V, p) takes values into the compact set $K = [0, V^*(1)] \times [0, 1]$. This means that $\mathcal{M}$ is **bounded** on K and thus, there is no problem to use the Optional Stopping Theorem**.** We look for a martingale $\mathcal{M}$ of the form $\mathcal{M}(t) = M(V(t), p(t)) + t$, where M is a smooth function satisfying the following boundary condition $M(V^*(p), p) = 0$. *Given the martingale property of* $\mathcal{M}$, we have $E^P\left[\mathcal{M}(\tau)\right] = \mathcal{M}(0)$ since τ^* is a stopping time. Then, the *boundary condition* $M(V^*(p), p) = 0$ implies

$$E^P\left[\tau^*\right] = T(V, p) = M(V, p).$$

Notice it is well known that when μ is constant, then

$$E(\tau^*) = \frac{\ln\left(\frac{V^*}{V}\right)}{\mu - \frac{\sigma^2}{2}}.$$

Hence, we must have

$$T(V,1)=\frac{\ln\left(\frac{V^*(1)}{V}\right)}{h-\frac{\sigma^2}{2}} \text{ and } T(V,0)=\frac{\ln\left(\frac{V^*(0)}{V}\right)}{l-\frac{\sigma^2}{2}}.$$

Using Ito lemma, one necessary condition for $\mathcal{M}$ to be a martingale is[2]

$$\begin{aligned}0 &= 1+V(ph+(1-p)l)M_1(V,p)+\frac{\sigma^2}{2}V^2M_{11}(V,p)\\ &\quad +V(h-l)p(1-p)M_{12}(V,p)+\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2(p(1-p))^2M_{22}(V,p).\end{aligned}$$

The proof is complete. ■

5.5. Appendix 5

Step 1: Derivation of the optimal investment frontier

Proof. We have seen that

$$F(V,p)=\sup_{\tau\geq 0}\frac{1}{\xi_{p,l}(0)}E^l\left[\xi_{p,l}(\tau)(V(\tau)-I)e^{-r\tau}\mid \mathcal{F}_0\right].$$

Recall that $\phi_{p,l}(t)=\frac{p(t)}{1-p(t)}$. To save notation, set $\phi_{p,l}=\phi$. Then define a new value function H such that

$$H(V,\phi)=(1+\phi)F(V,\frac{\phi}{1+\phi}).$$

Now define a new stochastic process $\theta=\phi V^{-\frac{h-l}{\sigma^2}}$. Recall that $d\phi(t)=\frac{h-l}{\sigma}\phi(t)dw(t)$, so by Ito lemma, we obtain

$$\begin{aligned}d\theta(t) &= V^{-\frac{h-l}{\sigma^2}}d\phi(t)-\frac{h-l}{\sigma^2}\phi V^{-\frac{h-l}{\sigma^2}-1}dV(t)-\frac{(h-l)^2}{\sigma^2}\phi V^{-\frac{h-l}{\sigma^2}}dt+\frac{1}{2}(h-l)(\frac{h-l}{\sigma^2}+1)\theta dt\\ &= \frac{h-l}{\sigma}\theta dw(t)-\frac{h-l}{\sigma^2}\theta(ldt+\sigma dw(t))-\frac{(h-l)^2}{2\sigma^2}\theta dt+\frac{1}{2}(h-l)\theta dt\\ &= \frac{(h-l)}{2\sigma^2}\theta\left(-2l-(h-l)+\sigma^2\right)dt\\ &= 0 \text{ since } h+l=\sigma^2.\end{aligned}$$

[2] To be more formal, this is just imposing the infinitesimal generator of $\mathcal{M}$ to be equal to 0.

Thus θ is actually a constant process. Then, define a new value function L for the variables (V, θ) such that

$$L(V, \theta) = H(V, \theta V^{\frac{h-l}{\sigma^2}}).$$

It follows that in the inaction region, L must satisfy the following PDE

$$rL(V, \theta) = lVL_1(V, \theta) + \frac{\sigma^2}{2}V^2 L_{11}(V, \theta).$$

The general solution is given by

$$L(V, \theta) = A(\theta)V^{\beta_l} + B(\theta)V^{\delta},$$

where A and B are smooth functions to be determined and β_l and δ are respectively the positive and negative root of the quadratic

$$\frac{\sigma^2}{2}x^2 + \left(l - \frac{\sigma^2}{2}\right)x - r = 0.$$

Then, the initial condition $L(0, \theta) = 0$ implies $B = 0$. Hence,

$$H(V, \phi) = A(\phi V^{-\frac{h-l}{\sigma^2}})V^{\beta_l}.$$

The smooth-pasting conditions for variables (V, ϕ) are

$$\begin{aligned} H(V^*(\phi), \phi) &= (1+\phi)(V^*(\phi) - I) \\ H_1(V^*(\phi), \phi) &= (1+\phi) \\ H_2(V^*(\phi), \phi) &= V^*(\phi) - I. \end{aligned}$$

The conditions are

$$\begin{aligned} A(\phi V^*(\phi)^{-\frac{h-l}{\sigma^2}})V^*(\phi)^{\beta_l} &= (1+\phi)(V^*(\phi) - I) \\ -\frac{h-l}{\sigma^2}\phi V^*(\phi)^{-\frac{h-l}{\sigma^2}}A'(\phi V^*(\phi)^{-\frac{h-l}{\sigma^2}})V^*(\phi)^{\beta_l} + \beta_l A(\phi V^*(\phi)^{-\frac{h-l}{\sigma^2}})V^*(\phi)^{\beta_l} &= (1+\phi)V^*(\phi) \\ V^*(\phi)^{-\frac{h-l}{\sigma^2}}A'(\phi V^*(\phi)^{-\frac{h-l}{\sigma^2}})V^*(\phi)^{\beta_l} &= V^*(\phi) - I. \end{aligned}$$

Eliminating the function A among the previous equations leads to

$$V^*(\phi) = \frac{\left(\beta_l(1+\phi) - \frac{h-l}{\sigma^2}\phi\right)I}{(\beta_l - 1)(1+\phi) - \frac{h-l}{\sigma^2}\phi}.$$

Since $\phi = \frac{p}{1-p}$, we ultimately obtain

$$\begin{aligned} V^*(p) &= (1 + \frac{1}{\beta_l - 1 - \frac{h-l}{\sigma^2}p})I \\ &= (1 + \frac{1}{p\beta_h + (1-p)\beta_l - 1})I. \blacksquare \end{aligned}$$

Step 2: Existence of the inverse function S

Proof. We want to show that

$$\begin{array}{rlcl} & (0,1) & \longrightarrow & \mathbb{R}_{++} \\ \Phi: & p & \longmapsto & \frac{p}{1-p} V^*(p)^{-\frac{h-l}{\sigma^2}} \end{array}$$

is a strictly increasing function and therefore admits an inverse denoted S. It is equivalent to show that $\Psi = \ln \Phi$ is strictly increasing from $-\infty$ to $+\infty$. Recall that $V^*(p) = (1 + \frac{1}{\beta_l - 1 - \frac{h-l}{\sigma^2}p})I$ so

$$\Psi(p) = \ln p - \ln(1-p) - \frac{h-l}{\sigma^2}\ln I - \frac{h-l}{\sigma^2}\ln(1 + \frac{1}{\beta_l - 1 - \frac{h-l}{\sigma^2}p}).$$

Differentiating with respect to p leads to

$$\Psi'(p) = \frac{1}{p(1-p)} - \left(\frac{h-l}{\sigma^2}\right)^2 \frac{1}{(\beta_l - 1 - \frac{h-l}{\sigma^2}p)(\beta_l - \frac{h-l}{\sigma^2}p)}.$$

In order to show that $\Psi'(p)$ is positive, it is enough to show that

$$g(p) = 2\left(\frac{h-l}{\sigma^2}\right)^2 p^2 - \frac{h-l}{\sigma^2}\left(\frac{h-l}{\sigma^2} + 2\beta_l - 1\right)p + \beta_l(\beta_l - 1) > 0 \text{ for all } p \in [0,1].$$

Note that $g(0) = \beta_l(\beta_l - 1) > 0$ and $g(1) = (\beta_l - 1 - \frac{h-l}{\sigma^2})(\beta_l - \frac{h-l}{\sigma^2}) = \beta_h(\beta_h - 1) > 0$. Differentiating g with respect to p leads to

$$g'(p) = 4\left(\frac{h-l}{\sigma^2}\right)^2 p - \frac{h-l}{\sigma^2}\left(\frac{h-l}{\sigma^2} + 2\beta_l - 1\right).$$

Thus, g' starts decreasing until $p^* = \frac{1}{4}\frac{\sigma^2}{h-l}\left(\frac{h-l}{\sigma^2} + 2\beta_l - 1\right)$. Then we show that

$$\frac{1}{4}\frac{\sigma^2}{h-l}\left(\frac{h-l}{\sigma^2} + 2\beta_l - 1\right) \geq 1.$$

The condition is equivalent to

$$2\beta_l \geq 3\frac{h-l}{\sigma^2} + 1.$$

Since $2\beta_l = 1 - \frac{2l}{\sigma^2} + \sqrt{\left(1 - \frac{2l}{\sigma^2}\right)^2 + \frac{8r}{\sigma^2}}$, the previous condition is equivalent to

$$\left(1 - \frac{2l}{\sigma^2}\right)^2 + \frac{8r}{\sigma^2} \geq \left(\frac{3h-l}{\sigma^2}\right)^2$$

and using the equality $h + l = \sigma^2$

$$(h-l)^2 + 8r(h+l) \geq (3h-l)^2$$

which implies

$$8r(h+l) \geq 8h^2 - 4hl.$$

This condition is met since $r > h > l$. Hence, on the interval $[0,1]$, g is decreasing and since $g(1) > 0$, g is positive on $[0,1]$. The desired result follows easily. ■

Step 3: Derivation of the Option Value

Proof. From the matching condition $F(V^*(p), p) = V^*(p) - I$, we obtain that

$$A\left(\frac{p}{1-p}V^*(p)^{-\frac{h-l}{\sigma^2}}\right)V^*(p)^{\beta_l} = \frac{V^*(p) - I}{1-p}.$$

Using the inverse function S, it follows that

$$A(x) = \frac{1}{1-S(x)}\left(\left(\frac{x(1-S(x))}{S(x)}\right)^{-\frac{\sigma^2}{h-l}} - I\right)\left(\frac{x(1-S(x))}{S(x)}\right)^{\frac{\sigma^2\beta_l}{h-l}} \quad \blacksquare$$

Step 4: Derivation of the average time to invest

Proof. Define a function K such that $K(V,\phi) = (1+\phi)T(V, \frac{\phi}{1+\phi})$. Using the PDE satisfied by T, it is easy to check that K must satisfy the following PDE

$$\begin{aligned} -(1+\phi) &= lVK_1(V,\phi) + \frac{\sigma^2}{2}V^2K_{11}(V,\phi) \\ &\quad + V(h-l)\phi K_{12}(V,\phi) + \frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2\phi^2K_{22}(V,\phi). \end{aligned}$$

The general solution of this equation is given by

$$K(V,\phi) = -\left(\frac{1}{l-\frac{\sigma^2}{2}} + \frac{\phi}{h-\frac{\sigma^2}{2}}\right)\ln V + C(\phi V^{-\frac{h-l}{\sigma^2}}) + D(\phi V^{-\frac{h-l}{\sigma^2}})V^{1-\frac{2l}{\sigma^2}}$$

where C and D are arbitrary smooth functions. Since $1-\frac{2l}{\sigma^2} = \frac{h-l}{\sigma^2}$, it follows that

$$T(V,p) = -\left(\frac{1-p}{l-\frac{\sigma^2}{2}} + \frac{p}{h-\frac{\sigma^2}{2}}\right)\ln V + (1-p)C(\frac{p}{1-p}V^{-\frac{h-l}{\sigma^2}}) + pE(\frac{p}{1-p}V^{-\frac{h-l}{\sigma^2}}),$$

where E is an arbitrary function such that $E(x) = D(x)/x$. Now suppose that when V hits a lower boundary $\underline{V}$, then $T(\underline{V},p) = 0$. The function T corresponding to our problem will be obtained by taking the limit when $\underline{V}$ goes to 0. We have the two following conditions

$$\begin{aligned}
C(\frac{p}{1-p}V^*(p)^{-\frac{h-l}{\sigma^2}}) + \frac{p}{1-p}E(\frac{p}{1-p}V^*(p)^{-\frac{h-l}{\sigma^2}}) &= \left(\frac{1}{l-\frac{\sigma^2}{2}} + \frac{1}{h-\frac{\sigma^2}{2}}\frac{p}{1-p}\right)\ln V^*(p) \\
C(\frac{p}{1-p}\underline{V}^{-\frac{h-l}{\sigma^2}}) + \frac{p}{1-p}E(\frac{p}{1-p}\underline{V}^{-\frac{h-l}{\sigma^2}}) &= \left(\frac{1}{l-\frac{\sigma^2}{2}} + \frac{1}{h-\frac{\sigma^2}{2}}\frac{p}{1-p}\right)\ln \underline{V}
\end{aligned}$$

Now set $u = \frac{p}{1-p}\underline{V}^{-\frac{h-l}{\sigma^2}}$ and using the inverse function S it follows for (x,u)

$$\begin{aligned}
C(x) + \frac{S(x)}{1-S(x)}E(x) &= \left(\frac{1}{l-\frac{\sigma^2}{2}} + \frac{1}{h-\frac{\sigma^2}{2}}\frac{S(x)}{1-S(x)}\right)\ln V^*(S(x)) \\
C(u) + u\underline{V}^{\frac{h-l}{\sigma^2}}E(u) &= \left(\frac{1}{l-\frac{\sigma^2}{2}} + \frac{u\underline{V}^{\frac{h-l}{\sigma^2}}}{h-\frac{\sigma^2}{2}}\right)\ln \underline{V}
\end{aligned}$$

For $u = x$, we obtain

$$\begin{aligned}
C(x) + S(x)E(x) &= \left(\frac{1}{l-\frac{\sigma^2}{2}} + \frac{S(x)}{h-\frac{\sigma^2}{2}}\right)\ln V(S(x)) \\
\left(\frac{S(x)}{1-S(x)} - x\underline{V}^{\frac{h-l}{\sigma^2}}\right)E(x) &= \frac{1}{h-\frac{\sigma^2}{2}}\left(\frac{S(x)}{1-S(x)}\ln V(S(x)) - x\underline{V}^{\frac{h-l}{\sigma^2}}\ln \underline{V}\right).
\end{aligned}$$

Taking the limit when $\underline{V}$ goes to 0 yields

$$\begin{aligned} C(x) &= \frac{\ln V^*(S(x))}{l - \frac{\sigma^2}{2}} \\ E(x) &= \frac{\ln V^*(S(x))}{h - \frac{\sigma^2}{2}}. \end{aligned}$$

Hence,

$$T(V, p) = \left(\frac{1-p}{l - \frac{\sigma^2}{2}} + \frac{p}{h - \frac{\sigma^2}{2}} \right) \ln \left(\frac{V^*(S(\frac{p}{1-p} V^{-\frac{h-l}{\sigma^2}}))}{V} \right). \blacksquare$$

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Tables

TABLE 1
Optimal trigger investment frontiers

σ^2		0.1			0.3			0.8	
	V_{NU}	V^*	Δ	V_{NU}	V^*	Δ	V_{NU}	V^*	Δ
p									
0	1.564	1.564	0	1.871	1.871	0	2.457	2.457	0
0.1	1.654	1.604	0.031	1.972	1.923	0.025	2.591	2.572	0.007
0.2	1.761	1.667	0.056	2.091	2.012	0.040	2.748	2.706	0.015
0.3	1.888	1.748	0.080	2.237	2.116	0.055	2.931	2.865	0.023
0.4	2.041	1.851	0.102	2.402	2.247	0.069	3.15	3.057	0.030
0.5	2.227	2.001	0.113	2.608	2.404	0.085	3.414	3.293	0.037
0.6	2.456	2.209	0.112	2.863	2.614	0.095	3.739	3.590	0.042
0.7	2.746	2.474	0.110	3.183	2.902	0.097	4.147	3.975	0.043
0.8	3.120	2.820	0.107	3.600	3.300	0.093	4.675	4.495	0.040
0.9	3.621	3.377	0.072	4.156	3.890	0.068	5.380	5.235	0.028
1	4.325	4.325	0	4.940	4.940	0	6.372	6.372	0

$r = 0.8,\ h = 0.6,\ l = 0.2,\ I = 1.$

TABLE 2

Average times until investment T^*, T_{NU}

$V = 0.49$

p	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
T^*	26.83	25.16	23.31	21.33	19.25	16.81	14.30	12.04	9.75	7.43	5.14
T_{NU}	26.83	15.49	11.18	8.93	7.58	6.70	6.09	5.68	5.40	5.22	5.14

$V = 1.71$

p	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
T^*	1.77	2.35	2.86	3.23	3.55	3.74	3.80	3.69	3.42	3.00	2.35
T_{NU}	1.77	1.57	1.54	1.56	1.61	1.68	1.77	1.88	2.01	2.16	2.35

$V = 2.83$

p	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
T^*	0	0	0	0	0	0	0	0.27	0.82	1.17	1.23
T_{NU}	0	0	0	0	0	0	0.03	0.35	0.64	0.93	1.23

$r = 0.8,\ h = 0.6,\ l = 0.2,\ \sigma^2 = 0.3,\ I = 1$

Footnotes

1. This is due to the fact that the unobservable drift is assumed to be constant.
2. To be more formal, this is just imposing the infinitesimal generator of $\mathcal{M}$ to be equal to 0.